On the maximum errors of polynomial approximations defined by interpolation and by least squares criteria

By M. J. D. Powell*

A function $f(x)$ is to be approximated by a polynomial of degree $n$ or less over the interval $a < x < b$. It is proved that the maximum errors of approximations defined by interpolation and by least squares criteria are within factors, independent of $f(x)$, of the least maximum error that can be achieved. Expressions for these factors are given; they are evaluated for approximations obtained by truncating the expansion of $f(x)$ in Chebyshev polynomials and for approximations obtained by interpolation at the zeros of a Chebyshev polynomial. The resultant numbers are not large: for example, if interpolation is used to evaluate a polynomial approximation of degree 20 it may be guaranteed that the resultant maximum error does not exceed the minimax error by more than a factor of four.

1. Introduction

A common reason for wishing to approximate $f(x)$ is that many values of the function are required, but the evaluation of each one is laborious. In this case it may prove economic to use function values to obtain an approximation to replace $f(x)$ in the main calculation.

Often the approximating function is chosen to be a polynomial, and we suppose that this choice has been made. The suitability of polynomials for this purpose is not questioned in this paper, rather we present and exploit theorems comparing the maxima of the error functions arising from approximations defined by interpolation, least squares and minimax criteria.

There is an excellent motive for preferring the polynomial approximation, $p^+(x)$ say, obtained by interpolating $/(x)$ at $(n + 1)$ prescribed values of the variable; calculating it requires the least number of evaluations of $f(x)$. However, straightforward interpolation is seldom used, perhaps because the well-known Runge phenomenon (see Todd, 1962, for instance) has led to the belief that the interpolation process may introduce extravagant errors. We shall establish that this is not the case for practical values of $n$ (say $n < 1000$) if the interpolation points are chosen to be the zeros of a Chebyshev polynomial.

Least squares methods (Clenshaw and Hayes, 1965) are probably the most popular. The criterion depends on a non-negative weight function, $\omega(x)$, the required approximation being $p^\omega(x)$, where

$$\int_a^b \omega(x)[f(x) - p^\omega(x)]^2dx \leq \int_a^b \omega(x)[f(x) - p(x)]^2dx,$$  

or, in notation defined by Clenshaw and Hayes (1965),

$$h^+ \leq h^\omega \leq h^*.$$  

that the redundancy allows higher order polynomial approximations to be obtained without additional values of $f(x)$, but we stress that, if $f(x)$ is calculated precisely, it is not reasonable to expect a least squares approximation to be substantially more accurate (accuracy being measured by the maximum value of the error function) than one obtained by interpolation at the recommended points.

The minimax approximation, $p^*(x)$, is defined by

$$\max_{a < x < b} |f(x) - p^*(x)| \leq \max_{a < x < b} |f(x) - p(x)|,$$

where $p(x)$ is any polynomial of degree $n$. Calculating it (Fraser, 1965) is an iterative process, and one must balance extra evaluations of $f(x)$ with the reduction they may cause in the maximum error of the current approximation. The theorems bound the gains that may be obtained by preferring a minimax approximation to one defined by least squares or interpolation.

I am indebted to Drs. D. Kershaw and W. Cheney for the following observations. Theorem 2 is a corollary of the theorem numbered 4.5.1 in Alexits (1961). Equation (25) relates $u(n)$ to the Lebesgue constant, several properties of which are given by Hardy (1942). Luttmann and Rivlin (1965) conjecture that $v(n)$, defined by (35), is obtained for $\theta = 0$; we prove the conjecture.

2. The theorems

We use the notation $x_0, x_1, \ldots, x_n$ for the interpolation points defining $p^+(x)$. We define the error functions

$$e^+(x) = f(x) - p^+(x)$$

$$e^\omega(x) = f(x) - p^\omega(x)$$

$$e^*(x) = f(x) - p^*(x),$$

and reserve $h^+$, $h^\omega$ and $h^*$ for the maximum values of $|e^+(x)|, |e^\omega(x)|$ and $|e^*(x)|, a < x < b$. Therefore

$$h^+ \geq h^*$$

$$h^\omega \geq h^*.$$
Polynomial approximation

**Theorem 1**

\[ h^+/h^* \leq 1 + \max_{a \leq x \leq b} \sum_{i=0}^{n} |l_i(x)| \] (5)

where \( l_i(x) \) is the polynomial of degree \( n \) satisfying

\[ l_i(x_j) = \delta_{ij}, \quad j = 0, 1, \ldots, n. \] (6)

**Proof**

\[ e^*(x) - e^+(x) = p^+(x) - p^*(x), \] (7)

which is a polynomial of degree \( n \) or less satisfying

\[ e^*(x_j) - e^+(x_j) = e^*(x_j), \quad j = 0, 1, \ldots, n. \] (8)

Therefore

\[ e^*(x) - e^+(x) = \sum_{i=0}^{n} e^*(x_i) l_i(x), \] (9)

from which we obtain

\[ |e^+(x)| = |e^*(x) - \sum_{i=0}^{n} e^*(x_i) l_i(x)| \]
\[ \leq |e^*(x)| + \sum_{i=0}^{n} |e^*(x_i)| |l_i(x)| \]
\[ \leq h^* \left\{ 1 + \sum_{i=0}^{n} |l_i(x)| \right\}. \] (10)

Since (10) holds for \( a \leq x \leq b \), the theorem follows.

**Theorem 2**

\[ h^*/h^* \leq 1 + \max_{a \leq x \leq b} \int_{a}^{b} \omega(y) \left| \sum_{i=0}^{n} \phi_i(x) \phi_i(y) \right| dy, \] (11)

where \( \phi_0(x), \phi_1(x), \ldots \) is the sequence of orthonormal polynomials satisfying

\[ \int_{a}^{b} \omega(x) \phi_i(x) \phi_j(x) dx = \delta_{ij}. \] (12)

**Proof**

We may write

\[ e^*(x) - e^o(x) = \sum_{i=0}^{n} \alpha_i \phi_i(x), \] (13)

where the numbers \( \alpha_0, \alpha_1, \ldots, \alpha_n \) are to be determined. Since \( e^o(x) \) is the error function of the least squares approximation,

\[ \int_{a}^{b} \omega(x) e^o(x) \phi_i(x) dx = 0, \quad i = 0, 1, \ldots, n. \] (14)

Therefore, using (12), (13) and (14),

\[ \alpha_i = \int_{a}^{b} \omega(y) e^o(y) \phi_i(y) dy. \] (15)

Hence

\[ |e^o(x)| = \left| e^*(x) - \sum_{i=0}^{n} \int_{a}^{b} \omega(y) e^*(y) \phi_i(y) \phi_i(x) dy \right| \]
\[ = \left| e^*(x) - \int_{a}^{b} \omega(y) e^*(y) \sum_{i=0}^{n} \phi_i(x) \phi_i(y) dy \right|. \]

\[ \leq |e^*(x)| + \int_{a}^{b} \omega(y) |e^*(y)| \left| \sum_{i=0}^{n} \phi_i(x) \phi_i(y) \right| dy \]
\[ \leq h^* \left\{ 1 + \int_{a}^{b} \omega(y) \left| \sum_{i=0}^{n} \phi_i(x) \phi_i(y) \right| dy \right\}. \] (16)

Again the inequality holds for \( a \leq x \leq b \), which proves Theorem 2.

Note that an alternative form of (11) may be obtained by using the identity (Szegö, 1939)

\[ \sum_{i=0}^{n} \phi_i(x) \phi_i(y) = \frac{k_n}{k_{n+1}} \phi_{n+1}(x) \phi_{n+1}(y) \frac{(x - y)}{(x - y)}, \] (17)

where \( k_n \) is the coefficient of \( x^n \) in \( \phi_n(x) \). (17) aids the numerical evaluation of (11) because, on account of the modulus signs, it is expedient to divide the range of integration at the zeros of the integrand. These zeros are the values of \( y \) satisfying

\[ \frac{\phi_{n+1}(y)}{\phi_n(y)} = \frac{\phi_{n+1}(x)}{\phi_n(x)}, \quad x \neq y. \] (18)

It may be established that the theorems provide least upper bounds on \( h^+/h^* \) and \( h^*/h^* \) in the event that the function \( f(x) \) may be any bounded square-integrable function defined on \( a \leq x \leq b \).

3. Application to truncated Chebyshev expansions

In this case the range of approximation is \(-1 < x < 1, \)

\[ \omega(x) = (1 - x^2)^{-1/2}, \] (19)

\[ \phi_0(x) = 1/\sqrt{\pi}, \] (20)

and

\[ \phi_t(x) = \sqrt{2/\pi} \cos(t\theta) = \sqrt{2/\pi} \cos(t\theta), \quad t = 1, 2, \ldots \] (21)

Hence the bound provided by Theorem 2 is

\[ u(n) = 1 + \max_{0 \leq \theta \leq \pi} \frac{1}{2 \pi} \int_{0}^{\pi} \sin \left( (n + \frac{1}{2}) \psi \right) d\psi, \] (22)

where the prime on the summation sign indicates that the first term is to be halved. Now

\[ \sin \left( (n + \frac{1}{2}) \theta + \psi \right) \]
\[ \sin \left( (n + \frac{1}{2}) \theta - \psi \right), \quad \psi = \frac{\psi}{2} \] (23)

so

\[ u(n) = 1 + \max_{0 \leq \theta \leq \pi} \frac{1}{2 \pi} \int_{0}^{\pi} \sin \left( (n + \frac{1}{2}) \theta + \psi \right) \sin \left( (n + \frac{1}{2}) \theta - \psi \right) d\psi \]
\[ + \frac{\sin \left( (n + \frac{1}{2}) \theta - (\theta - \psi) \right)}{\sin \left( (n + \frac{1}{2}) \theta - (\theta - \psi) \right)} \]
\[ = 1 + \max_{0 \leq \theta \leq 2\pi} \frac{1}{2 \pi} \int_{0}^{\pi} \sin \left( (n + \frac{1}{2}) \theta + \psi \right) \sin \left( (n + \frac{1}{2}) \theta - \psi \right) d\psi \]
Polynomial approximation

\[ \leq 1 + \max_{0 \leq \theta \leq \pi} \frac{1}{4\pi} \int_{-\pi}^{\pi} \left| \frac{\sin ((n + \frac{1}{2})(\theta + \psi))}{\sin (\frac{\theta}{4} + \psi)} \right| \left| \frac{\sin ((n + \frac{1}{2})(\theta - \psi))}{\sin (\frac{\theta}{4} - \psi)} \right| d\psi \]

\[ = 1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin ((n + \frac{1}{2})\psi)}{\sin (\frac{\psi}{4})} \right| d\psi. \]  

(24)

As this upper bound is attained when \( \theta = 0 \),

\[ u(n) = 1 + \frac{1}{2\pi} \int_{0}^{\pi} \left| \frac{\sin ((n + \frac{1}{2})\psi)}{\sin (\frac{\psi}{4})} \right| d\psi. \]  

(25)

In order to tabulate \( u(n) \) we re-employ (23), noting that in (25) the zeros of the integrand occur at

\[ \psi_j = \frac{j\pi}{(n + \frac{1}{2})}, \quad j = 1, 2, \ldots, n. \]  

(26)

Therefore, defining \( \psi_0 = 0 \) and \( \psi_{n+1} = \pi \),

\[ u(n) = 1 + \frac{2}{\pi} \sum_{j=1}^{n} (-1)^j \int_{\psi_j}^{\psi_{j+1}} \cos (t\psi) d\psi \]

\[ = 1 + \frac{2}{\pi} \sum_{j=1}^{n} (-1)^j \left[ \frac{1}{4\psi} + \sum_{t=1}^{n} \frac{\sin (t\psi)}{t!} \right]^{\psi_{j+1}}_{\psi_j} \]

\[ = 1 + \frac{1}{2n+1} - \frac{4}{\pi} \sum_{j=1}^{n} \frac{1}{j} \sum_{t=1}^{n} (-1)^j \sin (t\psi_j). \]  

(27)

Now

\[ \sum_{j=1}^{n} (-1)^j \sin (t\psi_j) = \sum_{j=1}^{n} \sin \left\{ j \left( \pi + \frac{\pi t}{n + \frac{1}{2}} \right) \right\} \]  

(28)

and

\[ \sum_{j=1}^{n} \sin (j\phi) = \frac{\cos (\frac{1}{2}\phi) - \cos ((n + \frac{1}{2})\phi)}{2 \sin (\frac{1}{2}\phi)}. \]  

(29)

Hence

\[ u(n) = \left( \frac{2n+2}{2n+1} \right) + \frac{2}{\pi} \sum_{t=1}^{n} \frac{1}{t} \tan \left( \frac{\pi t}{2n+1} \right). \]  

(30)

Using (30) values of \( u(n) \) were calculated; they are presented in the second column of Table 1.

It is apparent that the asymptotic dependence is logarithmic, and from (25) we may derive

\[ u(n) \sim \frac{4}{\pi^2} \log n. \]  

(31)

4. Application to interpolation at the zeros of a Chebyshev polynomial

We apply Theorem 1 to bound \( h^+ / h^* \) in the event that the range of the approximation is \(-1 < x < 1\) and the interpolation points are

\[ x_j = \cos \left( \frac{(j + \frac{1}{2})\pi}{n + 1} \right), \quad j = 0, 1, \ldots, n. \]  

(32)

Table 1

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<th>( n )</th>
<th>( u(n) )</th>
<th>( v(n) )</th>
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<td>1000</td>
<td>5.070</td>
<td>6.361</td>
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</table>

\( l_i(x) \) is derived by appealing to the identity, valid for \( i, j = 0, 1, \ldots, n \),

\[ \sum_{i=0}^{n} \cos \left( \frac{(i + \frac{1}{2})\pi}{n + 1} \right) \cos \left( \frac{(j + \frac{1}{2})\pi}{n + 1} \right) = \frac{n+1}{2} \delta_{ij}, \]  

(33)

from which it follows that

\[ l_i(\cos \theta) = \frac{2}{n+1} \sum_{i=0}^{n} \cos \left( \frac{(i + \frac{1}{2})\pi}{n + 1} \right) \cos (\theta t). \]  

(34)

Hence the required bound is

\[ v(n) = 1 + \frac{2}{n+1} \max_{0 \leq \theta \leq \pi} \sum_{i=0}^{n} \left| \sum_{t=0}^{n} \sin \left( \frac{(n + \frac{1}{2})\theta}{n + 1} \right) \cos (\theta t) \right|. \]  

(35)

Using (23) again,

\[ v(n) = 1 + \frac{1}{2(n+1)} \max_{0 \leq \theta \leq \pi} \sum_{i=0}^{n} \frac{\sin \left( (n + \frac{1}{2})\theta + \theta_i \right)}{\sin \left( \frac{1}{4}\theta + \theta_i \right)} \]

\[ + \frac{\sin \left( (n + \frac{1}{2})\theta - \theta_i \right)}{\sin \left( \frac{1}{4}\theta - \theta_i \right)} \]  

(36)

where

\[ \theta_i = (i + \frac{1}{2})\pi/(n + 1). \]  

(37)

Since

\[ \sin \left( (n + \frac{1}{2})\theta \pm \theta_i \right) \]

\[ = \sin \left( (n + 1)\theta \pm \frac{1}{4}(\theta \pm \theta_i) \right) \]

\[ = \pm (-1)^i \cos \left( (n + 1)\theta - \frac{1}{4}(\theta \pm \theta_i) \right), \]  

(38)

we may obtain from (36)

\[ v(n) = 1 + \frac{1}{2(n+1)} \max_{0 \leq \theta \leq \pi} \sin(\theta). \]  

(39)
where
\[ \chi(\theta) = | \cos ((n + 1)\theta) | \sum_{i=0}^{n} | \cot \left( \frac{1}{2}(\theta - \theta_i) \right) | - \cot \left( \frac{1}{2}(\theta - \theta_i) \right). \] (40)

We now establish that \( \chi(\theta) \) attains its maximum value at \( \theta = 0 \).

Consider the effect on \( \chi(\theta) \) of adding or subtracting an integral multiple of \( \frac{\pi}{n+1} \) to \( \theta \). Only the terms under the summation sign of (40) are altered, and the definition of \( \theta_i \) is such that the same \( 2(n + 1) \) cotangents occur, the change in \( \theta \) causing them to be paired differently. Since \( \chi(\theta) \) is an even function of \( \theta \) and since if
\[ 0 < \theta < \frac{\pi}{2(n+1)} \] (41)
both \( \cot(\theta + \theta_i) \) and \( \cot(\theta - \theta_i) \) are in the interval \([0, \frac{\pi}{n+1}]\) for \( i = 0, 1, \ldots, n \), we deduce that the required maximum value of \( \chi(\theta) \) is attained subject to (41). Now the zeros of \( l_n(\cos \theta) \) are all real, and they all occur in the interval
\[ \frac{\pi}{2(n+1)} < \theta < \pi. \] (42)

Consequently, within the interval (41) each term
\[ \left| \sum_{i=0}^{n} \cos \left( \frac{1}{n+1}(i+\frac{1}{2})\pi \right) \cos(\theta) \right| \] (43)
of (35) is monotonic decreasing. We conclude
\[ \max_{0 < \theta < \pi} \chi(\theta) = \chi(0) = 2 \sum_{i=0}^{n} \cot \left( \frac{1}{2}(i+\frac{1}{2})\pi \right). \] (44)

Therefore the required bound is
\[ v(n) = 1 + \frac{1}{n+1} \sum_{i=0}^{n} \tan \left( \frac{1}{2}(i+\frac{1}{2})\pi \right); \] (45)
it is tabulated in the third column of Table 1. The asymptotic dependence on \( n \) is
\[ v(n) \sim 2 \pi \log n. \] (46)

5. Remarks

We have not lost generality in supposing, in Sections 3 and 4, that the range of approximation is \( -1 < x < 1 \). If the range is \( a < x < b \) it may be established, through the transformation
\[ t = (a + b - 2x)/(a - b), \] (47)
that the third column of Table 1 is appropriate to the interpolation points
\[ x_j = \frac{1}{2} \left( (a + b) + (b - a) \cos \left( \frac{(j+\frac{1}{2})\pi}{n+1} \right) \right), \] (48)
for \( j = 0, 1, \ldots, n \). The basis of Theorems 1 and 2 is that the difference between the approximations that are being compared is a polynomial whose coefficients are determined from the properties of the approximations. This device has several applications.

The Chebyshev weighting function (19) is not optimal if the criterion is adopted that \( \omega(x) \) is to be chosen to minimize (11); indeed the optimal \( \omega(x) \) may depend on \( n \). Golomb (1965) asserts that no choice of weighting function can improve the asymptotic dependence given by (31) by more than a factor of two.

This paper is not directly relevant to deciding whether a polynomial of degree \( n \) can provide an adequate representation of \( f(x) \). Rather we have established that if the polynomial obtained by interpolation at the zeros of a Chebyshev polynomial is not satisfactory, it is unlikely that the minimax polynomial approximation of degree \( n \) is acceptable.

6. Acknowledgements

I am indebted to Mr. A. R. Curtis for many profitable discussions and to Mr. M. J. Hopper for programming the calculation of Table 1.

References


407