Overview

- Illustrative example
- Wavelets
- Homogenisation using Wavelets
- Pertinent example
Model problem

Consider the heat problem:

\[-(au')' = f \quad x \in \Omega = (0, 1)\]
\[u = 0 \quad x \in \partial \Omega\]

The weak formulation is:

\[\langle au', v' \rangle = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega)\]

Let us consider a heterogenous medium: \(a(x) = (2 + \cos(2\pi x))^{-1}\) and constant heat source: \(f(x) = 1\). In the following \(\epsilon = 2^{-7}\).
Solution \( u(x) = (1 - x)x + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x/\epsilon) \)

\[
u_1(x, x/\epsilon) = \frac{1}{2\pi} \sin(2\pi x/\epsilon)(\frac{1}{2} - x)\]

\[
u_2(x/\epsilon) = \frac{1}{4\pi^2} (1 - \cos(2\pi x/\epsilon))\]

\[
a(x) = (2 + \cos(2\pi \frac{x}{\epsilon}))^{-1}, \text{ on average } \overline{a(x)^{-1}} = 2. \text{ Analytically:} \]

\[
\hat{a} = \left( \int \frac{1}{a(x, y)} dy \right)^{-1} = \{\text{in this case}\} = \frac{1}{2}\]
Using 5 quadrature points: Relative $L^2$-norm and relative Energy-norm:

<table>
<thead>
<tr>
<th>N</th>
<th>Rel-L2</th>
<th>Rel-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.33750</td>
<td>0.49149</td>
</tr>
<tr>
<td>8</td>
<td>0.02265</td>
<td>0.35367</td>
</tr>
<tr>
<td>16</td>
<td>0.07265</td>
<td>0.34604</td>
</tr>
<tr>
<td>32</td>
<td>0.38711</td>
<td>0.49585</td>
</tr>
<tr>
<td>64</td>
<td>0.22329</td>
<td>0.39446</td>
</tr>
<tr>
<td>128</td>
<td>0.15300</td>
<td>0.36330</td>
</tr>
<tr>
<td>256</td>
<td>0.13390</td>
<td>0.35650</td>
</tr>
<tr>
<td>512</td>
<td>0.02572</td>
<td>0.14737</td>
</tr>
<tr>
<td>1024</td>
<td>0.00673</td>
<td>0.07521</td>
</tr>
<tr>
<td>2048</td>
<td>0.00171</td>
<td>0.03774</td>
</tr>
</tbody>
</table>
Using 63 quadrature points: Relative $L^2$-norm and relative Energy-norm:

<table>
<thead>
<tr>
<th>N</th>
<th>Rel-L2</th>
<th>Rel-E</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.18186</td>
<td>0.42617</td>
</tr>
<tr>
<td>8</td>
<td>0.14774</td>
<td>0.37587</td>
</tr>
<tr>
<td>16</td>
<td>0.13625</td>
<td>0.36107</td>
</tr>
<tr>
<td>32</td>
<td>0.13470</td>
<td>0.35770</td>
</tr>
<tr>
<td>64</td>
<td>0.13417</td>
<td>0.35680</td>
</tr>
<tr>
<td>128</td>
<td>0.13403</td>
<td>0.35658</td>
</tr>
<tr>
<td>256</td>
<td>0.13400</td>
<td>0.35650</td>
</tr>
<tr>
<td>512</td>
<td>0.02573</td>
<td>0.14738</td>
</tr>
<tr>
<td>1024</td>
<td>0.00673</td>
<td>0.07553</td>
</tr>
<tr>
<td>2048</td>
<td>0.00171</td>
<td>0.03850</td>
</tr>
</tbody>
</table>
Why the bad approximations?

FEM:

\[ \langle au'_h, v' \rangle = \langle f, v \rangle \quad \forall v \in V \subset H^1_0(\Omega) \]

\[ ||u_h - u||_V = \inf_{v \in V} ||v - u||_V \]

For coarse \( h \), there is no good approximation in energy norm.

\[ u(x) = x(1 - x) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x/\epsilon) \]

\( \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x/\epsilon) \) negligible in \( L^2 \)-norm

\( \epsilon u_1(x, x/\epsilon) \) NOT negligible in energy norm since

\[ \partial_x \epsilon u_1(x, x/\epsilon) \approx \partial_2 u_1(x, x/\epsilon) = \cos(2\pi x/\epsilon)(1/2 - x) \]
Some repetition of Wavelets

Wavelet spaces defined via shape function and mother wavelet:

\[ \phi_0(x) \]  \hspace{2cm} \[ \psi_0(x) \]

\[ \phi_{j,k} = 2^{j/2} \phi(2^j x - k) \]  \hspace{2cm} \[ \psi_{j,k} = 2^{j/2} \psi(2^j x - k) \]

\[ \text{span}(\{\phi_{j,k}\}_{k}) = V_j \subset V_{j+1} \]  \hspace{2cm} \[ \text{span}(\{\psi_{j,k}\}_{k}) = W_j \]

Orthonormal basis

\[ V_{j+1} = V_j \oplus W_j \]  \hspace{2cm} \[ V_j \perp W_j \]

\[ \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}) \]  \hspace{2cm} \[ \bigoplus_{j \in \mathbb{Z}} W_j = L^2(\mathbb{R}) \]
We consider the Haar Wavelet:
Wavelet based homogenisation

\[ V_n \cong \mathbb{R}^{2^n} \]

Abuse of notation: vector and function will have same symbol, i.e. \( U_j \in \mathbb{R}^{2^n} \) and \( U_j \in V_n \)

Given \( U_j \) we can extract the coarse and fine scale

\[ \mathcal{W}_j \in \mathbb{R}^{2^{n+1} \times 2^{n+1}} \text{ s.t. } \]

\[ \mathcal{W}_j U_{j+1} = \begin{bmatrix} U^f_j \\ U^c_j \end{bmatrix} \quad U^f_j \in W_j \quad U^c_j \in V_j \]

\( \mathcal{W}_j \) is orthonormal and sparse
FEM to Wavelet space

\[ U_{FEM} = 2^{-j/2} U_j \]
\[ L_{FEM} U = F_{FEM} \]
\[ L_{FEM}(2^{-j/2} U) = (2^{-j/2} F_{FEM}) \]

Idea: discretise in \( V_{j+1} \) but solve for \( U_j \in V_j \)

\[ LU_{j+1} = F \]
\[ \mathcal{W}_j LU_{j+1} = \mathcal{W}_j F \]
\[ \mathcal{W}_j L \mathcal{W}_j^T (\mathcal{W}_j U) = \mathcal{W}_j F \]

\[
\begin{bmatrix}
A_j & B_j \\
C_j & L_j
\end{bmatrix}
\begin{bmatrix}
U^f \\
U^c
\end{bmatrix} =
\begin{bmatrix}
F^f \\
F^c
\end{bmatrix}
\]

\[ U^f = A_j^{-1}(F^f - B_j U^c) \], if \( F^f = 0 \):

\[ (L_j - C_j A_j^{-1} B_j) U^c = F^c \]
Homogenised problem: \((L_j - C_jA_j^{-1}B_j)U^c = F^c\)

- Iterate to obtain homogenised operator in \(V_{j-1}\).
- \(A_j\) is sparse but its inverse may be dense.
- If \(A\) diagonally dominant, symmetric and tridiagonal, then
  \[|(A^{-1})_{kl}| \leq C\rho^{|k-l|}, \quad 0 < \rho < 1\]

- Do without most of \(A^{-1}\), e.g. truncate after a certain diagonal.

Decay rates exist for other types of matrices.
Asymptotic regime entered at $N = 1024 = 2^{10}$. Using the homogenised operator we should get good approximation using only $N = 512$. Using wavelet transformation one can extract all coarser solutions:

<table>
<thead>
<tr>
<th>$N$</th>
<th>L2 Hom.</th>
<th>E Hom.</th>
<th>L2</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.084683</td>
<td>0.397303</td>
<td>0.18186</td>
<td>0.42617</td>
</tr>
<tr>
<td>8</td>
<td>0.027246</td>
<td>0.35213</td>
<td>0.14774</td>
<td>0.37587</td>
</tr>
<tr>
<td>16</td>
<td>0.01188</td>
<td>0.33842</td>
<td>0.13625</td>
<td>0.36107</td>
</tr>
<tr>
<td>32</td>
<td>0.00808</td>
<td>0.33472</td>
<td>0.13470</td>
<td>0.35770</td>
</tr>
<tr>
<td>256</td>
<td>0.00678</td>
<td>0.17844</td>
<td>0.13400</td>
<td>0.35650</td>
</tr>
<tr>
<td>512</td>
<td>0.00674</td>
<td>0.14693</td>
<td>0.02573</td>
<td>0.14738</td>
</tr>
<tr>
<td>1024</td>
<td>0.00673</td>
<td>0.07553</td>
<td>0.02573</td>
<td>0.14738</td>
</tr>
</tbody>
</table>
A more suitable test case for wavelet homogenisation

\[ a(x) = \begin{cases} 
0.01 & x_0 - \epsilon \leq x \leq x_0 + \epsilon \\
1 & \text{else} 
\end{cases} \]

\[ \epsilon = 2^{-7} = 1/128 \]
Can be of considerable importance in higher dimensions. Suppose $\Omega = [0, 1]^d$, and that there is fine scale structure of size $\epsilon$.

$$N \geq \left( \frac{1}{\epsilon} \right)^d$$

If algorithm has complexity $n^3$:

$$\#\text{flops} \left[ \left( \frac{1}{\epsilon} \right)^d \right]^3$$
Thank You!