Approximation theory, Homework 3
Sparse reconstructions

Problem 1
Assume that for $\delta_2 < 1$, the inequalities
\[
(1 - \delta_2 s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_2 s)\|x\|_2^2 \quad \text{(RIP)}
\]
hold for all $2s$-sparse $x$. Let $b = Ax$ where $\hat{x}$ is $s$-sparse. Show that the sparsest solution to $b = Ax$ is $\hat{x}$. (RIP is an acronym for restricted isometry property, which is used in theorems for proving, e.g., stability of solutions in $\ell_1$ regularization methods.)

Problem 2
Let $y(t)$ be a complex sinusoid (in noise):
\[
y(t) = \sum_{\ell=1}^{s} \hat{x}_\ell \exp(i\hat{\theta}_\ell t).
\]
The goal is to reconstruct $s, \hat{x}_\ell, \hat{\theta}_\ell$ given a small number of measurement $y(t_1), \ldots, y(t_n)$. A procedure for this is as follows:

- Grid of frequency domain: $\theta_1, \ldots, \theta_m$.
- Set up the linear equation:
  \[
  \begin{pmatrix}
  y(t_1) \\
  \vdots \\
  y(t_n)
  \end{pmatrix} = \begin{pmatrix}
  \exp(i\theta_1 t_1) & \cdots & \exp(i\theta_m t_1) \\
  \vdots & \ddots & \vdots \\
  \exp(i\theta_1 t_n) & \cdots & \exp(i\theta_m t_n)
  \end{pmatrix} \begin{pmatrix}
  x_1 \\
  \vdots \\
  x_m
  \end{pmatrix}
  \Rightarrow y = Ax
  \]
- Reconstruct the solution from
  \[
  \min_x \|x\|_1 \text{ subject to } y = Ax.
  \]

Implement this reconstruction procedure in e.g., Matlab. An easy to use toolbox for solving convex optimization problems is CVX. Otherwise Matlab’s built-in solvers also works.

Consider cases where $s \ll n \ll m$, hence the system is underdetermined (e.g., $s = 10, n = 30, m = 100$). Test a couple of parameter sets $\{s, n, m\}$ and see if the reconstruction procedure works.

1. Find a set of parameters where the procedure works and display the results (don’t choose too easy problem).
3. Compare the solution with the Tychonov regularizer:
  \[
  \min_x \|x\|_2 \text{ subject to } y = Ax.
  \]