Program 2.9: Knapsack Approximation Scheme

input Set X of n items, for each $x_i \in X$, values p_i, a_i, positive integer b, rational number $r > 1$;
output Subset $Y \subseteq X$ such that $\sum_{x_i \in Y} a_i \leq b$;
begin
p_{max} := maximum among values p_i;
$t := \lfloor \log(\frac{r-1}{r} \frac{p_{max}}{n}) \rfloor$;
x' := instance with profits $p'_i = \lfloor p_i / 2^t \rfloor$;
Y := solution returned by Program 2.8 with input x';
return Y
end.

it is sufficient to observe that the execution of the body of both the first and the third **for** loop of Program 2.8 requires a constant number of steps. QED

The running time of Program 2.8 is polynomial in the values of the profits associated with the items of the instance of the problem. These values are exponential in the length of the input if we use any reasonable encoding scheme (in fact $\log p_i$ bits are sufficient to encode the value p_i) and for this reason the algorithm is not a polynomial-time one. However, in order to stress the fact that the running time is polynomial in the value of the profits, we will say that the running time of the algorithm is *pseudo-polynomial*.

Besides being interesting by itself, Program 2.8 can be used to obtain a polynomial-time algorithm that, given an instance x of MAXIMUM KNAPSACK and a bound on the desired performance ratio, returns a feasible solution of x whose quality is within the specified bound. The algorithm works in the following way: instead of directly solving the given instance x, it solves an instance x' which is obtained by scaling down all profits by a power of 2 (depending on the desired degree of approximation). Instance x' is then solved by Program 2.8 and from its optimal solution the approximate solution of the original instance x is finally derived. The algorithm is shown in Program 2.9: since the algorithm's behavior depends on the required performance ratio, it is called an *approximation scheme* for MAXIMUM KNAPSACK.

Theorem 2.18 ► *Given an instance x of* MAXIMUM KNAPSACK *with n items and a rational number $r > 1$, Program 2.9 returns a solution in time $O(rn^3/(r-1))$ whose measure $m_{AS}(x,r)$ satisfies the inequality $m^*(x)/m_{AS}(x,r) \leq r$.*

PROOF Let $Y(x,r)$ be the approximate solution computed by Program 2.9 with input x and r and let $Y^*(x)$ be an optimal solution of x, with measure $m^*(x)$. It is easy to see that, since for any item inserted in $Y(x,r)$ the largest error