when applied to an input graph G whose vertices are considered in the order $(v_1, v_2, \ldots, v_n)$. *Then, the following inequality holds:*

$$k_n \leq 1 + \max_{1 \leq i \leq n} \min(d_n(v_i), i-1).$$

PROOF

If the algorithm does not introduce a new color to color vertex v_i, then $k_i = k_{i-1}$. Otherwise, $k_i = k_{i-1} + 1$ and the degree of v_i in G_i must satisfy the inequality

$$d_i(v_i) \geq k_{i-1}.$$

By induction on i, it thus follows that

$$k_n \leq 1 + \max_{1 \leq i \leq n} (d_i(v_i)). \tag{2.6}$$

Since $d_i(v_i)$ is clearly bounded both by $d_n(v_i)$ and by $i-1$ (that is, the number of other nodes in G_i), the theorem follows. QED

An immediate consequence of the above theorem is the following result.

◀ Corollary 2.11

For any ordering of the vertices, the sequential coloring algorithm uses at most $\Delta + 1$ *colors to color a graph G, where* Δ *denotes the highest degree of the vertices of G.*

Observe that a non-increasing ordering of the vertices with respect to their degree minimizes the upper bound in Theorem 2.10. Hence, we use this ordering to obtain the sequential algorithm called *Decreasing Degree*. Unfortunately, the number of colors used by this algorithm can be much larger than the number of colors used by the optimal solution. In fact, the following example shows that there exist 2-colorable graphs with $2n$ vertices and maximum degree $n-1$ for which *Decreasing Degree* could require n colors (as a side effect, the example also shows that the bound of Corollary 2.11 is tight).

◀ Example 2.7

Let n be an integer and $G(V,E)$ be a graph with $V = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and $E = \{(x_i, y_j) \mid i \neq j\}$ (see Fig. 2.5 where $n = 4$). Note that all vertices have the same degree $n-1$ and that G can be easily colored with 2 colors. However, if the initial ordering of the vertices is

$$(x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

then *Decreasing Degree* uses n colors.