cities in I_k, either c_r and c_s were adjacent also in I^* or, by the triangle inequality, there exists a path in I^* starting from c_r and ending at c_s of length at least $D(r,s)$. As a consequence, $m^*(x) \geq m_k^*$, for each k.

Since $D(i,j) \geq \min(l(c_i), l(c_j))$ for all pairs of cities c_i and c_j, by summing over all edges (c_i, c_j) that belong to I_k we obtain

$$m_k^* \geq \sum_{(c_i,c_j)\in I_k} \min(l(c_i), l(c_j)) = \sum_{c_i \in C_k} \alpha_i l(c_i),$$

where α_i is the number of cities $c_j \in C_k$ adjacent to c_i in I_k and such that $i > j$ (hence, $l(c_i) \leq l(c_j)$). Clearly, $\alpha_i \leq 2$ and $\sum_{c_i \in C_k} \alpha_i$ equals the number of cities in I_k. Since the number of cities in I_k is at most $2k$, we may derive a lower bound on the quantity $\sum_{c_i \in C_k} \alpha_i l(c_i)$ by assuming that $\alpha_i = 0$ for the k cities $c_1, \ldots, c_k$ with largest values $l(c_i)$ and that $\alpha_i = 2$ for all the other $|C_k| - k$ cities. Hence, we obtain Eq. (2.5) since

$$m^*(x) \geq m_k^* \geq \sum_{c_i \in C_k} \alpha_i l(c_i) \geq 2 \sum_{i=k+1}^{\min(2k,n)} l(c_i).$$

Summing all Eqs. (2.5) with $k = 2^j$, for $j = 0, 1, \ldots, \lceil \log n \rceil - 1$, we obtain that

$$\sum_{j=0}^{\lceil \log n \rceil - 1} m^*(x) \geq \sum_{j=0}^{\lceil \log n \rceil - 1} 2 \sum_{i=2^j+1}^{\min(2^{j+1},n)} l(c_i),$$

which results in

$$\lceil \log n \rceil m^*(x) \geq 2 \sum_{i=2}^{n} l(c_i).$$

Since, by hypothesis, $m^*(x) \geq 2l(c_1)$, the lemma is then proved. QED

◀ Theorem 2.5

For any instance x of MINIMUM METRIC TRAVELING SALESPERSON *with n cities, let $m_{NN}(x)$ be the length of the tour returned by* Nearest Neighbor *with input x. Then $m_{NN}(x)$ satisfies the following inequality:*

$$m_{NN}(x)/m^*(x) \leq \frac{1}{2}(\lceil \log n \rceil + 1).$$

PROOF

Let $c_{k_1}, c_{k_2}, \ldots, c_{k_n}$ be a tour resulting from the application of *Nearest Neighbor* to x. The proof consists in showing that, if we associate to each city c_{k_r} ($r = 1, \ldots, n$) the value $l(c_{k_r})$ corresponding to the length of edge $(c_{k_r}, c_{k_{r+1}})$ (if $r < n$) or of edge (c_{k_n}, c_{k_1}) (if $r = n$), then we obtain a mapping l that satisfies the hypothesis of Lemma 2.4. Since $\sum_{i=1}^{n} l(c_{k_i}) = m_{NN}(x)$, the theorem follows immediately by applying the lemma.