By adding Eq. (2.2) and twice Eq. (2.1) we obtain that

$$\sum_{i=1}^{m_{Gr}(G)} (d_i+1)^2 \leq n(2\delta+1).$$

The left-hand side of the above inequality is minimized when $d_i + 1 = n/m_{Gr}(G)$, for all i (this is an application of the Cauchy-Schwarz inequality; see Appendix A). It follows that

$$n(2\delta+1) \geq \sum_{i=1}^{m_{Gr}(G)} (d_i+1)^2 \geq \frac{n^2}{m_{Gr}(G)}.$$

QED Hence, $m_{Gr}(G) \geq n/(2\delta+1)$ and the theorem is proved.

The following theorem provides a relationship between the measure of the solution found by the greedy algorithm and the optimal measure.

Theorem 2.3 ► *Given a graph G with n vertices and m edges, let $\delta = m/n$. Program 2.2 finds an independent set of value $m_{Gr}(G)$ such that*

$$m^*(G)/m_{Gr}(G) \leq (\delta+1).$$

PROOF The proof is similar to that of the preceding theorem: in this case, we additionally count in Eq. (2.1) the number of edges that are incident to vertices of some optimal solution.

Namely, fix a maximum independent set V^* and let k_i be the number of vertices in V^* that are among the $d_i + 1$ vertices deleted in the i-th iteration of the **while** loop of Program 2.2.

Clearly, we have that

$$\sum_{i=1}^{m_{Gr}(G)} k_i = |V^*| = m^*(G). \tag{2.3}$$

Since the greedy algorithm selects the vertex with minimum degree, the sum of the degree of the deleted vertices is at least $d_i(d_i + 1)$. Since an edge cannot have both its endpoints in V^*, it follows that the number of deleted edges is at least $(d_i(d_i+1) + k_i(k_i - 1))/2$.

Hence we can improve Eq. (2.1) to obtain

$$\sum_{i=1}^{m_{Gr}(G)} \frac{d_i(d_i+1)+k_i(k_i-1)}{2} \leq \delta n. \tag{2.4}$$

Adding Eqs. (2.2), (2.3) and twice (2.4), we obtain the following bound: