or, equivalently,

$$\frac{\arccos(\mathbf{y}_i^* \cdot \mathbf{y}_j^*)}{\pi} \geq \frac{\beta}{2}(1 - (\mathbf{y}_i^* \cdot \mathbf{y}_j^*)).$$

Since $QP\text{-}CUT(x)$ is a relaxation of $IQP\text{-}CUT(x)$, we have that

$$\begin{aligned} \mathrm{E}[m_{RWC}(x)] &\geq \frac{1}{2}\beta \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - \mathbf{y}_i^* \cdot \mathbf{y}_j^*) \\ &= \beta m^*_{QP-CUT}(x) \geq \beta m^*_{IQP-CUT}(x) = \beta m^*(x), \end{aligned}$$

where $m_{RWC}(x)$ is the measure of the solution returned by Program 5.3.

Since it is possible to show that $\beta > 0.8785$ (see Exercise 5.10), the Lemma is thus proved. QED

Regarding the time complexity of Program 5.3, it is clear that the algorithm runs in polynomial time if and only if it is possible to solve $QP\text{-}CUT(x)$ in polynomial time. Unfortunately, it is not known whether this is possible. However, the definition of $QP\text{-}CUT(x)$ can be slightly modified in order to make it efficiently solvable: the modification simply consists in considering variables $\mathbf{y}_i$ as vectors in the n-dimensional space instead that in the 2-dimensional one. In particular, the n-dimensional version of $QP\text{-}CUT(x)$ is defined as

$$\begin{array}{ll} \texttt{maximize} & \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - \mathbf{y}_i \cdot \mathbf{y}_j) \\ \texttt{subject to} & \mathbf{y}_i \in S_n \qquad\qquad 1 \leq i \leq n, \end{array}$$

where S_n denotes the n-dimensional unit sphere. Observe that, clearly, the above analysis of the expected performance ratio of Program 5.3 can still be carried out if we refer to this new version of $QP\text{-}CUT(x)$.

In order to justify this modification, we need some definitions and results from linear algebra. First of all, we say that a $n \times n$ matrix M is *positive semidefinite* if, for every vector $x \in R^n$, $x^T M x \geq 0$. It is known that a $n \times n$ symmetric matrix M is positive semidefinite if and only if there exists a matrix P such that $M = P^T P$, where P is an $m \times n$ matrix for some $m \leq n$. Moreover, if M is positive semidefinite, then matrix P can be computed in polynomial time (see Bibliographical notes).

Observe now that, given n vectors $\mathbf{y}_1, \ldots, \mathbf{y}_n \in S_n$, the matrix M defined as $M_{i,j} = \mathbf{y}_i \cdot \mathbf{y}_j$ is positive semidefinite. On the other hand, from the above properties of positive semidefinite matrices, it follows that, given a $n \times n$ positive semidefinite matrix M such that $M_{i,i} = 1$ for $i = 1, \ldots, n$, it is