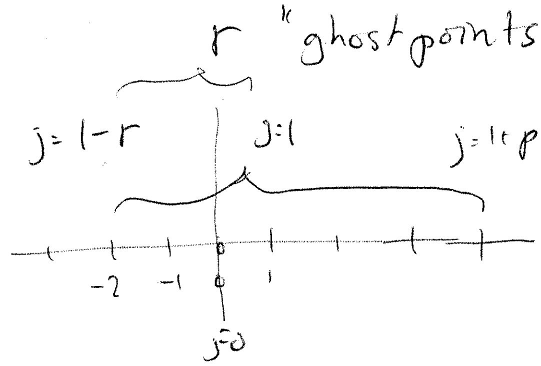
Normal mode analysis, fully discrete case

Analysis similar to the semi-discrete case with the changes "Laplace transform" → "Z-transform" and "$\operatorname{Re} s \gtreqless 0$" → "$|z| \gtreqless 1$".

Consider constant coefficient, 1/4-plane problems:

$$(*)\quad \begin{cases} u_j^{n+1} = Q u_j^n + k F_j^n & j = 1, 2, \dots \\ B u^n = g^n, \quad \|u\| < \infty \\ u_j^0 = f_j \end{cases}$$



where

$$Q = \sum_{m=-r}^{p} a_m E^m$$

$a_m = a_m(h, k)$ bounded for k/h fixed.

$$B u^n \sim \quad u_\nu^n = \sum_{j=1}^{q} b_{\nu,j} u_j^n + g_\nu^n + \sum_{j=1}^{q} b'_{\nu,j} u_j^{n-1} + \dots \qquad \nu = 0, \dots, -r+1$$

May also contain different timelevels $(n, n-1, \dots)$

Ex:

$$\begin{cases} u_j^{n+1} = \dfrac{u_{j-1}^n + u_{j+1}^n}{2} + \dfrac{k}{h}\left(\dfrac{u_{j+1}^n - u_{j-1}^n}{2}\right) & \text{Lax-Friedrichs} \\ u_0^n - 2u_1 + u_2 = 0 & (\text{or } u_0^n = g^n) \\ u_j^0 = f_j \end{cases}$$

for $u_t = u_x$.

As in semidiscrete case, we analyze:

a) $g=0$, $f\neq 0$, $F=0$

b) $g\neq 0$, $f=0$, $F=0$.

$\Rightarrow$ Full solution is a sum of these.

<u>Case a)</u> $g=0$, $f\neq 0$.

Suppose $Q\phi_j = z\phi_j$, $B\phi_j = 0$, $\|\phi\|<\infty$

$\Rightarrow$ Then

$$u_j^n = z^n \phi_j$$

is a solution of $(*)$ (Check: $Qu_j^n = z^n Q\phi = z^{n+1}\phi = u_j^{n+1}$ + BC is ok.)

with initial data $f_j = \phi_j$.

$$\Rightarrow \quad \|u^n\| = |z|^n \|\phi_j\| = |z|^{t_n/k} \|\phi\|$$

This cannot be bounded by $Ke^{\alpha t_n}$ independent of k, if $|z|>1$.

Godunov Ryabenkii condition for fully discrete case:

If z is an eigenvalue of (Q, B) then $|z|\leq 1$ is necessary for stability.

Case b) $g \neq 0,\ f = 0$.

Use (unilateral) Z-transform for a sequence $\{u^n\}$:

$$\hat{u}(z) = \sum_{n=0}^{\infty} z^{-n} u^n \qquad (|z| \text{ [redacted] large enough to get convergence})$$

Since $u^n \sim e^{\alpha t_n} = (e^{\alpha k})^n$ we need $|z| > e^{\alpha k}$.

Then:

1) $u^n = \frac{1}{2\pi i} \oint_C \hat{u}(z) z^{n-1} dz$ (inversion) $C = \{|z| = R\}$ with R large enough. $(> e^{\alpha k})$

2) $\widehat{u^{n+1}} = z\hat{u}(z) - z \cdot u^0$

3) $\sum |u^n|^2 = \frac{1}{2\pi} \int_{|z|=1} |\hat{u}(z)|^2 dz$ (Parseval)

Z-transform of (*) with $f = F = 0$:

$$(a)\ \begin{cases} z\hat{u}_j(z) = Q\hat{u}_j(z) \\ \vec{B}\hat{u}_j(z) = \vec{g}(z) \ (c), \quad \|\vec{u}(z)\| < \infty \ (b) \end{cases}$$

Same eigenvalue problem as before!

Need to be able to solve it with $|z|$ large enough, to use inversion formula.

We can solve this in the same way [redacted] as in semi discrete case:

1) (a) is a difference equation parameterized by z.

If $Q = \sum_{m=-r}^{p} a_m E^m$, then the general solution is

$$\hat{u}_j(z) = \sum_{m=1}^{r+p} \sigma_m \kappa_m^j(z)$$

where $\kappa_m(z)$ are roots of the characteristic polynomial:

$$a_{-r} + \kappa a_{-r+1} + \cdots \kappa^r(a_0 - z) + \cdots + \kappa^{r+p} a_p = P_z(\kappa).$$

σ_m are free parameters.

2) Boundary cond. (b) $\|\hat{u}\| < \infty$

Need $\sigma_m = 0$ for $|\kappa_m| \geq 1$. to satisfy (b).

- Theorem: If von Neumann cond. satisfied for periodic problem, then:
  1) No roots κ_m with $|\kappa_m| = 1$ if $|z| > 1$.
  2) Precisely r roots with $|\kappa_m| < 1$ if $|z| > 1$.

$\Rightarrow$ For $|z| > 1$ the solution is of the form

$$(**) \quad \hat{u}_j(z) = \sum_{m=1}^{r} \sigma_m \kappa_m^j(z)$$

$\leftarrow$ note

if periodic problem is stable.

3) Boundary condition (c) $\vec{B}\vec{u} = \vec{g}$.

Enter the form $(**)$ into boundary conditions:

$\nu = 0, \ldots, r+1$

$$\hat{u}_\nu^n = \sum b_{\nu j} u_j^n + \sum \hat{b}_{\nu j} u_j^{n-1} + g_\nu$$

$$\Rightarrow \quad \hat{u}_\nu(z) = \sum \left(b_{\nu j} \hat{u}_j(z) + \hat{b}_{\nu j} \frac{1}{z} \hat{u}_j(z) \right) + \hat{g}_\nu$$

$$\Rightarrow \quad C(z) \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_r \end{pmatrix} = \begin{pmatrix} \hat{g}_0 \\ \vdots \\ \hat{g}_{r+1} \end{pmatrix} \quad \text{where } C \in \mathbb{C}^{r \times r}$$

Same kind of $C(z)$ as before.

Need $\det C(z) \neq 0$ for $|z| > 1$. Otherwise there is a solution to eigenvalue problem with $\vec{g} = 0$.

$\therefore$ GR-condition $\Leftrightarrow$ $\det C(z) \neq 0$ for $|z| > 1$.

- Kreiss condition:

$$\det C(z) \neq 0 \quad \text{for} \quad |z| \geq 1.$$

(Equivalent:

$$|\hat{u}_j(z)| \leq K\, |\hat{g}(z)| \quad |z| \geq 1. \quad K \text{ indep of } z.)$$

- Together with energy estimate for full Cauchy problem ($j = -\infty \ldots \infty$), $\|u^{n+1}\| \leq \|u^n\|$ = stronger than stability !! and technical conditions it is <u>sufficient</u> for <u>strong stability.</u> (cf Theorem 2.9 in book.)

- Note $C(z)$ was defined with the roots κ_m such that $|\kappa_m| < 1$ for $|z| > 1$. When $|z| = 1$ κ_m may have $|\kappa_m| = 1$. and there may be more roots $|\kappa_{m'}(z)| = 1$. These latter ones are <u>not</u> part of $C(z)$.

Example

Lax – Friedrichs scheme:

$$\begin{cases} u_j^{n+1} = \dfrac{u_{j+1}^n + u_{j-1}^n}{2} + \dfrac{\lambda}{2}\left(u_{j+1}^n - u_{j-1}^n\right) & j = 1, \dots \\ u_2 - 2u_1 + u_0 = g \end{cases}$$

Z-transform $\Rightarrow$

(1) $$z\hat{u}_j = \frac{\lambda+1}{2}\hat{u}_{j+1} + \frac{-\lambda+1}{2}\hat{u}_{j-1}$$

Characteristic polynomial $\Rightarrow$

$$(\lambda+1)K^2 - 2zK + (1-\lambda) = 0$$

$$\Rightarrow \quad K_{1,2} = \frac{z \pm \sqrt{z^2+\lambda^2-1}}{1+\lambda}$$

Pick $|K|<1$ for $|z|>1$ (always possible by theorem since L-F stable)

$$K_1 = \frac{z - \sqrt{z^2+\lambda^2-1}}{1+\lambda}$$

$\therefore$ $\hat{u}_j = \sigma_1 K_1^j(z)$ satisfies (1) and $\|\hat{u}_j\| < \infty$ for $|z|>1$.

Boundary conditions:

$$\hat{u}_2 - 2\hat{u}_1 + \hat{u}_0 = \hat{g} \Rightarrow$$

$$\sigma(K_1^2 - 2K_1 + 1) = \hat{g} \Rightarrow \sigma(K-1)^2 = \hat{g}.$$

$\therefore \quad C(z) = (K(z) - 1)^2$ (Scalar here since $r = 1$)

$\det C(z) = 0 \iff K(z) = 1$.

Not possible for $|z| > 1$ by construction ($|K| < 1$)

How about for $|z| = 1$?

$K = 1 \Rightarrow (\lambda + 1) 1^2 - 2z \cdot 1 + 1 - \lambda = 0$

$\Rightarrow \quad z = 1$ only possibility

But $K_1(1) = \dfrac{1 - \sqrt{1 + \lambda^2 - 1}}{1 + \lambda} = \dfrac{1 - \lambda}{1 + \lambda} < 1$.

(i.e. $K_2(1) = 1$)
not K_1.

$\therefore \det C(z) \neq 0$ for $|z| \geq 1$. Kreiss condition is satisfied.

Remarks

- Multistep methods can be transformed to one-step methods for systems as before.

$$u^{n+1} = Q_0 u^n + Q_1 u^{n-1} + \dots + Q_s u^{n-s}$$

Let $v = \begin{pmatrix} u^n \\ \vdots \\ u^{n-s} \end{pmatrix}$.

Then

$$v^{n+1} = \begin{pmatrix} Q_0 & Q_1 & \dots & Q_s \\ I & 0 & & 0 \\ & I & & \\ & & \ddots & \\ & & I & 0 \end{pmatrix} v^n$$

- In higher dimensions one first Fourier transforms in transversal variables. Eg. in 2D.

$$\begin{cases} \dfrac{d u_{i,j}}{dt} = Q_x u_{ij} + Q_y u_{ij} \\ u_{0,j} - 2u_{1,j} + u_{2,j} = 0 \\ u_0 = f_j \end{cases}$$

Let $\hat{u}_i(\omega) = \sum u_{ij} e^{ij\omega_y}$ (periodic in y)

$$\begin{cases} \dfrac{d\hat{u}_i}{dt} = (\hat{Q}_x + \hat{Q}_y)\, \hat{u}_i \\ \hat{u}_0 - 2\hat{u}_1 + \hat{u}_2 = 0 \end{cases}$$

$\Rightarrow$ One ~~step~~ -dimensional problem for each $\underline{\omega_y}$..

All conditions must now hold <u>uniformly in ω_y</u>.

- For scalar problems

$$v_j^{n+1} = Q v_j^n \qquad Q = \sum_{m=-r}^{p} a_m E^m$$

$$v_j^0 = 0$$

$$v_\nu^{n+1} = g_\nu^{n+1} \qquad \nu = -r+1, \dots, 0$$

always satisfies the Kreiss condition. (Goldberg & Tadmor).

- Generalized stability.

Much theory developed via another stability concept:

"strongly stable in generalized sense"

$$\int_0^\infty e^{-\eta t}\|u(t)\|^2 dt \le K(\eta)\int_0^\infty e^{-\eta t}(\|F(t)\|^2 + |g(t)|^2)dt$$

$K(\eta)\to 0$ as $\eta\to\infty$

when initial data $f=0$. Natural for Laplace / Z-transform setting.

- Enough to prove convergence (except if initial error very rough.)
- Schemes can be stable in this sense without satisfying the Kreiss cond. or being boundary stable.

- Strip problem

When we have two boundaries $0<x<1$ we have in general that if each boundary satisfies Kreiss condition and we have a energy estimate then strip problem is stable (in same sense as the 1/4-problems are).

- Variable coefficients

Stability for all Frozen coefficient problems typically give stability also for variable coefficient problem. Hard to find precise statements of this though.