Num PDE

Introduction

- (Mainly linear), time-dependent PDEs, hyperbolic & parabolic: Model problems:

$$u_t = a u_{xx}, \quad u_t = a u_x, \quad u_{tt} = a u_{xx}$$

heat eq. transp. eq. wave eq.

- Finite difference methods.

$$x_j = j\,\cancel{\Delta x}^{\,h}, \quad t_n = n\,\cancel{\Delta t}^{\,k} \qquad u_j^n \approx u(t_n, x_j)$$

Ex: $$u_j^{n+1} = u_j^n + a \frac{\cancel{\Delta t}^{\,k}}{\cancel{\Delta x^2}^{\,h^2}} \left(u_{j-1}^n - 2u_j^n + u_{j+1}^n \right) = u_j^n + a\lambda D_+ D_- u_j^n$$

$$\begin{pmatrix} D_+ u_j = u_{j+1} - u_j \\ D_- u_j = u_j - u_{j-1} \end{pmatrix}$$

- Convergence $u_j^n \to u(t_n, x_j)$?

When, why, how fast ?

Different levels of difficulties/simplifications:

- Constant coefficients $\Rightarrow$ Variable coeff. $\Rightarrow$ nonlinear
  $a = a$ $\quad a = a(x)$ $\quad a = a(u)$
- Boundaries / no boundaries (all of $\mathbb{R}$/periodic)
- 1, 2, 3 dimensions
- Semi discrete, fully discrete
  $\frac{du_j}{dt} = Qu_j$ $\qquad \cancel{d}\, u_j^{n+1} = Q u_j^n$
- Scalar, systems, forcing, lower order terms, ...

Well-posedness of PPE

- Basic condition for numerical solvability..

$$(*)\quad \begin{cases} \dfrac{\partial u}{\partial t} = P(t,x,\partial_x)\,u + F(t,x) & x>0 \\ u(0,x) = f(x) \\ u(t,0) = g(t) \end{cases}$$

Def: (*) is (strongly) well-posed if (for all smooth enough f, g, F)

1) $\exists$ a unique solution u

2) stability:

$$\|u(\cdot,t)\|^2 \le K e^{\alpha t}\left(\|f\|^2 + \int_0^t \|F(\tau,\cdot)\|^2 + |g(\tau)|^2\,d\tau\right)$$

(several versions available)

(Here $\|f\|^2 = \int |f(x)|^2 dx$)

$\Rightarrow$ "small perturbations in data (f,g,F) $\Rightarrow$ small perturbation in solution (u)"

$$\begin{cases} \dfrac{\partial \hat u}{\partial t} = P(t,x,\partial_x)\hat u + F(t,x) + \delta_F(t,x) \\ \hat u(0,x) = f(x) + \delta_f(x) \\ \hat u(t,0) = g(x) + \delta_g(x) \end{cases}$$

$\Rightarrow$ $w = \hat u - u$ satisfies.

$$\begin{cases} \dfrac{\partial w}{\partial t} = P(t,x,\partial_x)\,w + \delta_F(t,x) \\ w(0,x) = \delta_f \\ w(t,0) = \delta_g \end{cases}$$

$$\Rightarrow \|\hat u - u\|^2 \le K e^{\alpha t}\left(\|\delta_f\|^2 + \int |\delta_F| + |\delta_g|\right)$$

Examples

- (Strongly) Hyperbolic systems

$$u_t = Au_x + Bu + C \qquad u(x,0) = f(x)$$

where A has $\underline{\text{real}}$ eigenvalues & is diagonalizable

(*) $\qquad A = S\Lambda S^{-1}$, Λ is diagonal (real)

Ex: wave equation ($u_{tt} = u_{xx}$) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, eig $= \pm 1$.

- Parabolic systems.

$$u_t = Au_{xx} + Bu_x + Cu + D$$

(**) where eigenvalues λ of A satisfy $\mathrm{Re}\,\lambda \geq \delta > 0$.

Variable coefficients: ($A = A(t,x)$, etc.)

Hyperbolic: (*) holds for all t, x and $H(t,x) = (S^{-1})^* \Lambda S^{-1}(t,x)$ depends smoothly on (t,x).

Parabolic: (**) holds for all (t,x), uniformly. (δ indep of x)

Higher dimensions:

Hyperbolic: $\quad u_t = A_1 u_{x_1} + \cdots + A_d u_{x_d}$

Let $P(\omega) := \omega_1 A_1 + \cdots + \omega_d A_d$ $\qquad \omega = (\omega_1, \dots, \omega_d)$

(*) holds for all ω, such that $|\omega|_2 = 1$ and $\|S(\omega)\| + \|S^{-1}(\omega)\| < K$ indep of ω.

Parabolic: $\quad u_t = A_1 u_{x_1 x_1} + \cdots + A_d u_{x_d x_d}$

Let $P(\omega) = \omega_1^2 A_1 + \cdots + \omega_d^2 A_d$

(**) holds for all ω with $|\omega|_2 = 1$ uniformly in ω.

Counter examples – ill posed problems

Backward heat equation:

$$u_t = -u_{xx} \qquad u(0,x) = f(x)$$

Unique solution exists but not well-posed. Also true for:

"Cauchy problem" for Laplace:

$$u_{tt} = -u_{xx} \qquad u(0,x) = f(x),\ u_t(0,x) = g(x)$$

Note: (This is the same as the Cauchy-Riemann equations

$z = u + iv \Rightarrow \quad u_x = -v_y \ , \ u_y = v_x \qquad \Leftrightarrow i z_x = z_y$

$z(x,y)$ analytic

up to choice of BC.)

Time stepping Helmholtz

$$u_{tt} = k^2 u - u_{xx} \qquad u(0,x) = f(x) \quad u_t(0,x) = g(x)$$

Convergence of difference schemes

Consider a general scheme:

$$\begin{cases} u_j^{n+1} = Q\, u_j^n + k F_j^n \\ u_j^0 = f_j \end{cases}$$

(no boundaries)

Ex:

$$u_t + u_x = F(x), \quad u(0,x) = f(x)$$

$$\Rightarrow \begin{cases} u_j^{n+1} = u_j^n - \frac{k}{h}\left(u_j^n - u_{j-1}^n\right) + k F(x_j) \\ u_j^0 = f(x_j) \end{cases}$$

In general

$$Q = Q(\lambda) = \sum_{m=-r}^{p} a_m E^m \qquad E u_j = u_{j+1}$$

$$\lambda = k/h$$

We want to prove $u_j^n \longrightarrow u(t_n, x_j)$ as $h, k \to 0$.

General strategy:

1) Show stability, i.e.

$$\|u^n\|_h^2 \leq K e^{\alpha t_n}\left(\|f\|_h^2 + \sum_{m=0}^{n-1} \|F^m\|_h^2 k\right)$$

where $\|u^n\|_h^2 = \sum h (u_j^n)^2$ etc.

C.f. stability part of well-posedness def.

2) Show consistency, i.e.

if $u(t,x)$ is exact sol. of PDE

$$u(t_{n+1}, x_j) = Q\, u(t_n, x_j) + kF_j + \underbrace{k\,\tau_j^n}_{\text{local truncation error}}$$

$$u(0, x_j) = f_j + \delta_f$$ – error in initial data, typically $= 0$.

If $\tau_j^n = \mathcal{O}(h^p + k^q) \Rightarrow$

scheme has order of accuracy p in space & q in time.

3) Lax equivalence theorem:

stability + consistency

$\Longleftrightarrow$

convergence

$\Rightarrow$ part easy:

Let $e_j^n = u(t_n, x_j) - u_j^n$. Then

$$\begin{cases} e_j^{n+1} = Q u(t_n, x_j) + kF_j + k\tau_j^n - Qu_j^n - kF_j \\ \quad\;\;\; = Q e_j^n + k\tau_j^n \\ e_j^0 = \delta_f \end{cases}$$

stability $\Rightarrow$ $\|e_j^n\| \le K e^{\alpha t_n}\left(\|\delta_f\|^2 + \sum \|\tau_j^n\|^2 k\right)$

$$\le C(T)\left(\|\delta_f\|^2 + \mathcal{O}(h^p + k^q)\right)$$

$\longrightarrow 0$ if $\delta_f = \mathcal{O}(h)$

- Stability usually the difficult part to show.
- Similar methods used to show FD stability and PDE well-posedness. We will ~~[illegible]~~ only talk about the former in class, but see book for relations to the latter.