

EXAMPLE 12.1. If we write Burgers' equation (12.1) in the quasilinear form

$$u_t + uu_x = 0 \quad (12.3)$$

then a natural finite difference method, obtained by a minor modification of the upwind method for $u_t + au_x = 0$ (and assuming $U_j^n \geq 0$ for all j, n) is

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n (U_j^n - U_{j-1}^n). \quad (12.4)$$

The method (12.4) is adequate for smooth solutions but will not, in general, converge to a discontinuous weak solution of Burgers' equation (12.1) as the grid is refined. Consider, for example, the data from (11.1), which in discrete form gives

$$U_j^0 = \begin{cases} 1 & j < 0 \\ 0 & j \geq 0. \end{cases} \quad (12.5)$$

Then it is easy to verify from (12.4) that $U_j^1 = U_j^0$ for all j . This happens in every successive step as well and so $U_j^n = U_j^0$ for all j and n , regardless of the step sizes k and h . As the grid is refined, the numerical solution thus converges very nicely to the function $u(x, t) = u_0(x)$. This is not a weak solution of (12.1) (or of (12.2) either).

In this example the solution is obviously wrong, but similar behavior is seen with other initial data that may give reasonable looking results that are incorrect. Figure 12.1 shows the true and computed solutions at time $t = 1$ with Riemann data $u_l = 1.2$ and $u_r = 0.4$. We get a nice looking solution propagating at entirely the wrong speed.

EXERCISE 12.1. Show that (12.4) is consistent with both (12.1) and (12.2).

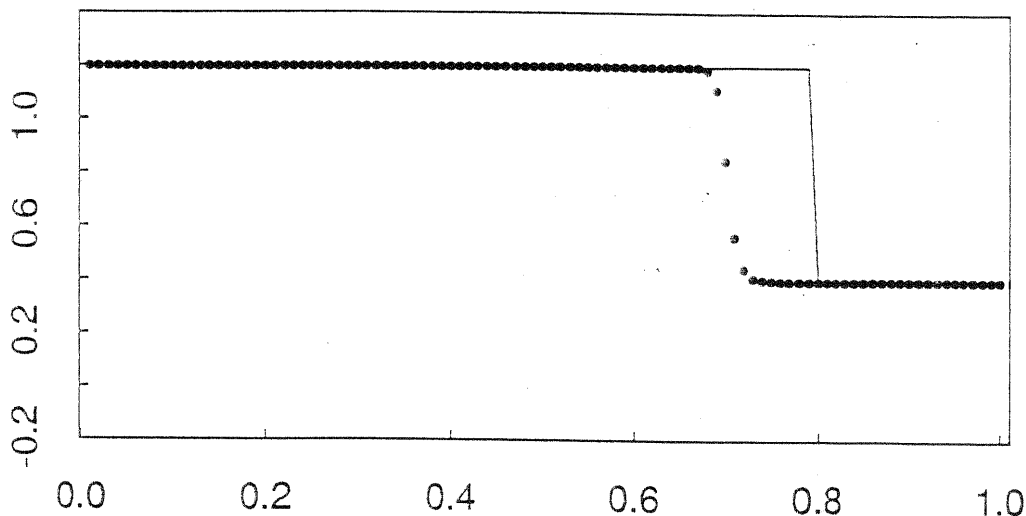


Figure 12.1. True and computed solutions to Burgers' equation using a nonconservative method.