KTH Computer Science
and Communication

Numerical Methods for Partial Differential Equations, 7.5 ECTS

Homework 3

1. Consider the central semi-discrete approximation of the transport equation $u_t = u_x$ in the right halfplane,

$$\begin{aligned}\frac{du_j}{dt} &= D_0 u_j, \qquad j = 1, 2, \ldots, \\ u_j(0) &= 0, \qquad j = 1, 2, \ldots, \\ ||u(t)||_h &< \infty.\end{aligned}$$

   Construct an *unstable* boundary condition at $j = 1$. Show theoretically that it is unstable and verify it numerically by solving the strip problem with $u_N = 0$ and some smooth non-zero initial data. (Note that the solution may be bad but still stable. Check that it grows when $h \to 0$.)

2. Consider the fourth order semi-discrete approximation for $u_t = u_x$,

$$\begin{aligned}\frac{du_j}{dt} &= \frac{4}{3}D_0(h)u_j - \frac{1}{3}D_0(2h)u_j := \frac{4}{3}\frac{u_{j+1} - u_{j-1}}{2h} - \frac{1}{3}\frac{u_{j+2} - u_{j-2}}{4h}, \qquad j = 1, 2, \ldots, \\ u_j(0) &= 0, \qquad j = -1, 0, 1, 2, \ldots, \\ u_{-1}(t) &= g_{-1}(t), \\ u_0(t) &= g_0(t), \\ ||u(t)||_h &< \infty.\end{aligned}$$

   (a) Show that the approximation satisfies the Kreiss condition.

   (b) Use the previous result and the energy method to derive the estimate

$$||u(t)||_h^2 \le C \int_0^t |g_{-1}(s)|^2 + |g_0(s)|^2 ds.$$

   Hint: Use (2.58) in the book and the relations $\langle u, D_0(h)u\rangle_{1,\infty} = -u_1 u_0/2$ and $\langle u, D_0(2h)u\rangle_{1,\infty} = -(u_{-1}u_1 + u_0 u_2)/4$.

   (c) Change the boundary conditions to

$$u_{-1}(t) = u_1(t), \qquad u_0(t) = u_2(t),$$

   and show that the Godunov–Ryabenkii condition is satisfied, but not the Kreiss condition.
   Hint: The characteristic polynomial has a root $\kappa = -1$ at $s = 0$.
   (If you have time: Check stability numerically, setting $u_N = u_{N+1} = 0$. This is a rather subtle exercise since the instability here is weak.)

These exercises are due Nov 24.