CONVERGENCE RATES FOR AN ADAPTIVE DUAL WEIGHTED RESIDUAL FINITE ELEMENT ALGORITHM

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Abstract. We study fundamental convergence properties of an adaptive algorithm based on the dual weighted residual error representation,

$$\text{error} = \sum_{\text{elements}} \text{error density} \times \text{mesh size}^{2+d},$$

applied to piecewise linear tensor finite element approximation of functionals of solutions to second order elliptic partial differential equations in bounded domains of $\mathbb{R}^d$. We show first that the optimal adaptive isotropic mesh uses a number of elements proportional to the $d/2$ power of the $L^{\frac{d}{d+2}}$ quasi-norm of the error density; the same error for approximation with a uniform mesh requires a number of elements proportional to the $d/2$ power of the larger L^1 norm of the same error density. The main result is then a proof that the adaptive algorithm based on successive subdivisions of elements reduces the maximal error indicator with a factor or stops with the error asymptotically bounded by the tolerance using the optimal number of elements, up to a problem independent factor. An important step is to prove that the error density based on localized averages of second order difference quotients of the primal and dual finite element solutions converge pointwise. The averages are used since the difference quotients itself do not converge pointwise for adapted meshes. The proof uses weak convergence techniques and a symmetrizer for the second order difference quotients.

Contents

1. Introduction to Adaptive Finite Element Algorithms

This work analyzes the convergence rate of an adaptive finite element algorithm to approximate functionals of the solution, $u : \Omega \to \mathbb{R}$, of the second order elliptic partial differential equation

$$-\text{div}(a\nabla u) = f \tag{1}$$

in an open bounded domain $\Omega \subset \mathbb{R}^d$ with boundary condition $u|_{\partial\Omega} = 0$. The paper presents, for simplicity, first the two dimensional and linear case, $d = 2$ and $a, f : \Omega \to \mathbb{R}$, which then directly generalizes to higher dimensions, $d > 2$. It is also easy to extend the study to some nonlinear problems, see Remark 2.8. This work uses some simplifying properties of linear tensor finite element approximation, but an extension to approximation by piecewise polynomials of degree $k > 1$ seems possible.

There are numerous studies on error estimates for adaptive finite element methods applied to partial differential equations, e.g., [1], [3, 4], [6], [7, 8], [16, 17], [18, 19], [23], and some work on the convergence of adaptive algorithms [5], [14], [24]. However, important numerical complexity theory, on how convergence rates for adaptive finite element algorithms depend on the computational work, is not as well developed, but there are recent contributions. The work [13] shows the efficiency of adaptive approximation of functions, including wavelet expansions, based on smoothness conditions in Besov spaces. Inspired by this approximation result, first the work [11] proves that a wavelet-based adaptive N-term approximation algorithm produces a solution with asymptotically optimal error in the energy norm for linear coercive elliptic problems. Then [9, 25] extend the ideas of [24] to prove similar optimal error estimates in the energy norm for piecewise linear elements applied to the Poisson equation. The modification includes a coarsening step in the adaptive algorithm to obtain bounds on the work.

Our work focuses on linear tensor finite element approximation of functionals of the solution to (1), inspired by [8], [16] and [17]. Section 2 uses the dual weighted residual method to derive an asymptotic expansion of this error, with computable leading order term $\sum_K \bar\rho_K h_K^{2+d}$ where h_K is the mesh size and $\bar\rho_K$ is the error density for element K. Section 3 applies this expansion to prove convergence rates, depending on the number of degrees of freedom, for an adaptive finite element algorithm.

What is the right measure of convergence rates for adaptive finite element algorithms applied to (1)? For a constant mesh size h, approximations with error $\mathcal{O}(h^p)$ require computational work with $\mathcal{O}(1/h^d)$ operations, using optimal multigrid solvers. The accuracy $\epsilon \equiv \mathcal{O}(h^p)$ is hence asymptotically determined by the number of elements $N = \mathcal{O}(1/h^d) = \mathcal{O}(\epsilon^{-d/p})$. This simple asymptotic complexity estimate, $\mathcal{O}(\epsilon^{-d/p})$, is one of the most basic and well used numerical analysis measures of the performance of approximations. Analogously, for adaptive methods, it seems natural to study the approximation error and the associated work, proportional to the number of elements, as the tolerance parameter tends to zero. For the second order accurate piecewise linear finite elements on a uniform mesh, the number of elements to reach a given approximation error turns out to be proportional to the $d/2$ power of the L^1-norm of the error density; this work shows that the smallest number of isotropic elements in an adaptive mesh is proportional to the $d/2$ power of the smaller $L^{\frac{d}{d+2}}$ quasi-norm of the same error density. These norms of the error density are therefore good measures of the convergence rates and define our optimal number of elements, explained in Section 3.

Section 3 constructs an algorithm which subdivides the elements with error indicators, $|\bar{\rho}_K|h_K^{2+d}$, greater than TOL/N and stops if all N elements have sufficiently small error indicators. In particular the algorithm has no coarsening step. Theorems 3.1, 3.4 and 3.5 in Section 3 prove that each refinement level of this adaptive algorithm decreases the maximal error indicator with a factor, less than 1, or stops with an error asymptotically bounded by TOL and with asymptotically optimal number of elements, N, in the finest mesh, up to a problem independent factor. The total number of elements, including all refinement levels, can be bounded by $\mathcal{O}(N)$, provided the tolerance in each refinement level decreases by a constant factor, see Theorem 3.8. Varying tolerance has the drawback that the final stopping tolerance is not a priori known; on the other hand, with constant tolerance, the total number of elements including all levels is bounded by the larger $\mathcal{O}(N \log N)$.

The reports [21] and [20, 26] introduced adaptive algorithms for weak approximation of ordinary and stochastic differential equations, respectively. Their extension to partial differential equations here is partly straightforward except for the pointwise convergence of the error density and a hanging node constraint: to prove convergence of the error density for approximation of ordinary differential equations is simple, while the corresponding convergence result for partial differential equations is hard, requiring structured adapted meshes and detailed analysis special to tensor finite elements. In fact the work [8] writes "The strategies for mesh adaption is largely based on heuristic grounds. One hard open problem is the rigorous proof of the convergence of local residual terms and weights to certain 'limits' ". Note that such pointwise convergence of the error density, based on second order difference quotients, is well known for structured uniform meshes; however Remark 2.2 below shows by an example that second order difference quotients of smooth functions do not in general converge pointwise for adapted meshes. To prove convergence of the second order difference quotients, in the error density, our proof instead uses localized averages and a symmetrizer.

2. Convergence of the Error Density

2.1. An Error Representation. The finite element approximation u_h, of u in (1), is based on the standard variational formulation in the function set V_h of continuous piecewise bilinear functions in $H^1_0(\Omega)$, using an adaptive quadrilateral mesh with hanging nodes cf. [8]. The Sobolev space $H^1_0(\Omega)$ is the usual Hilbert space of functions on Ω, vanishing on $\partial\Omega$, with bounded first derivatives in $L^2(\Omega)$. Let $\mathcal{T}$ denote the set of quadrilaterals K and let h_K be the local mesh size, i.e. the length of the longest edge of K. Assume we first want to compute a linear functional value $(u, F) \equiv \int_\Omega uF dx$ for a given function $F \in L^2(\Omega)$. Let $(\cdot, \cdot)$ denote the duality pairing in $H^{-1} \times H^1_0$, which reduces to the usual inner product in $L^2(\Omega)$ on $L^2 \times L^2$. Define the residual $\mathcal{R}(v) = -\mathrm{div}(a\nabla v) - f$ as a distribution in $H^{-1}(\Omega)$ for $v \in H^1_0(\Omega)$. Then the variational problems for $u \in H^1_0(\Omega)$ and $u_h \in V_h$ are

$$
\begin{aligned}
(\mathcal{R}(u), v) &= 0, \quad \forall v \in H^1_0(\Omega), \\
(\mathcal{R}(u_h), v) &= 0, \quad \forall v \in V_h. \qquad (2)
\end{aligned}
$$

Define the dual function $\varphi \in H^1_0(\Omega)$ by

$$
(a\nabla v, \nabla\varphi) = (F, v), \quad \forall v \in H^1_0(\Omega) \qquad (3)
$$

to obtain

$$(u - u_h, F) = (a\nabla(u - u_h), \nabla\varphi) = (\mathcal{R}(u_h), -\varphi).$$

The orthogonality (2) implies $(u - u_h, F) = (\mathcal{R}(u_h), v - \varphi)$ for all $v \in V_h$, and the choice $v = \pi\varphi \in V_h$, where π is the nodal interpolant on V_h, yields the error representation

$$(u - u_h, F) = (\mathcal{R}(u_h), \pi\varphi - \varphi)\,. \tag{4}$$

The global error (4) can be split into residual parts supported in the interior of quadrilaterals and on their edges

$$\begin{aligned} (u - u_h, F) = &\sum_{K\in\mathcal{T}} \int_K (-\mathrm{div}(a\nabla u_h) - f)(\pi\varphi - \varphi)dx \\ &+ \sum_{K\in\mathcal{T}} \int_{\partial K} a\frac{\partial u_h}{\partial n}\,(\pi\varphi - \varphi)\,ds, \end{aligned} \tag{5}$$

where n is the outward normal to the element K and $\partial/\partial n = n \cdot \nabla$.

2.2. **Approximation of the Error Density.** The goal in this section is to derive a computable approximation of the error representation (5). An adaptive algorithm providing a reliable error bound and efficient use of the degrees of freedom must use an error expansion

$$(u - u_h, F) \simeq \sum_K \bar\rho_K h_K^{2+d} \tag{6}$$

where the error density $\bar\rho$ is essentially independent of the mesh size, since the asymptotic error density is used to find the optimal mesh.

Precise analysis of the adaptive algorithms [21] for ordinary and stochastic differential equations [20, 22] was obtained by deriving convergence of an error density. This work generalizes those adaptive algorithms to partial differential equations. The main new ingredient is to prove convergence of the error density. For general meshes this convergence of the error density $\bar\rho$ does not hold, since the orientation of the elements varies. It may hold locally for successive refinements based on hanging nodes which generate parallel subsequent elements. The purpose here is to analyze the asymptotic behavior of the error density $\bar\rho$ for adaptive refinements, with parallel element edges. Therefore we restrict the study to adaptive hanging node meshes of parallel squares. The use of quadrilaterals and squares can directly be extended to higher dimensional cubes using tensor elements. The reason we first study tensor element meshes with hanging nodes is that they are simple in the sense that other refinements using e.g. subdivision of simplex, in three and higher dimensions cf. [15], generate new edges which are not parallel to the old and would require additional analysis.

To define the error density $\bar\rho$ we use the linear tensor finite element approximation $\varphi_h \in V_h$, of the dual function φ in (3), defined by

$$(a\nabla v, \nabla\varphi_h) = (F, v), \quad \forall v \in V_h. \tag{7}$$

Then one would like to use second order difference quotients of u_h and φ_h to approximate the error density. On uniform meshes the second difference quotients of u_h and φ_h converge and the proof uses the translation invariance of the mesh. However, non uniform adapted meshes are not translation invariant and Remark 2.2 shows that a second order difference quotient of the discrete functions u_h or φ_h

does not in general converge pointwise to the corresponding second order derivatives of u or φ, respectively. We solve this problem by using instead localized averages of second order difference quotients.

Consider a function w which is defined on a discretization of an interval $[0,a]$ with nodes $\{x_j : \ j = 0, \ldots, \bar{N}+1\} =: \bar{\mathcal{N}}$, where $x_0 = 0$ and $x_{\bar{N}+1} = a$. Let $h_+ \equiv x_{i+1} - x_i$ and $h_- \equiv x_i - x_{i-1}$ denote two consecutive edge sizes. Then define the average mesh size $\bar{h}$ and the second order difference

$$\begin{aligned} \bar{h}_i &\equiv \frac{h_+ + h_-}{2} = \frac{x_{i+1} - x_{i-1}}{2}, \\ D^2 w(x_i) &\equiv \frac{1}{\bar{h}_i}\left(\frac{w(x_i+h_+) - w(x_i)}{h_+} - \frac{w(x_i) - w(x_i - h_-)}{h_-}\right). \end{aligned} \tag{8}$$

The localized average is based on a non negative function, $\psi^{x_j} : \mathcal{N} \to \mathbb{R}$, $j = 1, \ldots, \bar{N}$ with $\psi_i^j \equiv \psi^{x_j}(x_i)$ and a positive parameter α, measuring the width of the average, where the averaging function satisfies

$$\psi_\cdot^j \geq 0, \tag{9}$$

$$\|\bar{h}.D^2\psi_\cdot^j\|_{\ell^1} = \mathcal{O}(\alpha^{-2}), \tag{10}$$

$$\psi_0^j + \psi_{\bar{N}+1}^j = \mathcal{O}(\alpha^{-1}), \tag{11}$$

and the weak convergence

$$\Big|\sum_{i=1}^{\bar{N}} \psi_i^j \bar{h}_i v(x_i) - v(x_j)\Big| = \mathcal{O}(\alpha), \quad \forall v \in \mathcal{C}^1(0,a). \tag{12}$$

Define the average difference

$$\overline{D^2 w}(x_j) \equiv \sum_{i=1}^{\bar{N}} \bar{h}_i D^2 w(x_i) \psi^j(x_i). \tag{13}$$

Section 2.3 presents an efficient method to compute such averages $Y \in \mathbb{R}^{\bar{N}+2}$ of $D^2 w \in \mathbb{R}^{\bar{N}}$ based on the equation

$$Y_i - \alpha^2 D^2 Y_i = D^2 w_i, \quad i = 1, \ldots, \bar{N}$$

with homogeneous Neumann boundary conditions, $Y_0 = Y_1$, $Y_{\bar{N}} = Y_{\bar{N}+1}$. The convergence proof requires α to be sufficiently large, slightly larger than the pointwise error $|u - u_h|$ and $|\varphi - \varphi_h|$.

Let us define $\bar{h}D_1^2 w$ and $\bar{h}D_2^2 w$ as the difference quotients $\bar{h}D^2 w$, in (8), with respect to the x_1 and x_2 directions, respectively. The approximate error density, $\overline{\rho}$, is now defined by

$$\begin{aligned} \overline{\rho}_K \equiv \frac{1}{24}[& 2(-\nabla a \cdot \nabla u_h - f)(\overline{D_1^2\varphi_h} + \overline{D_2^2\varphi_h}) \\ & - \left(a\overline{D_2^2 u_h}\ \overline{D_1^2\varphi_h}\right)_{e_{11}} - \left(a\overline{D_2^2 u_h}\ \overline{D_1^2\varphi_h}\right)_{e_{12}} \\ & - \left(a\overline{D_1^2 u_h}\ \overline{D_2^2\varphi_h}\right)_{e_{21}} - \left(a\overline{D_1^2 u_h}\ \overline{D_2^2\varphi_h}\right)_{e_{22}}] \end{aligned} \tag{14}$$

where e_{ij} denotes the j'th edge parallel to the x_i direction in the square K, illustrated in Figure 2.2.

Let $W^{1,\infty}$ denote the usual Sobolev space of functions with bounded first order derivatives in $L^\infty(\Omega)$. Our main result in this section is

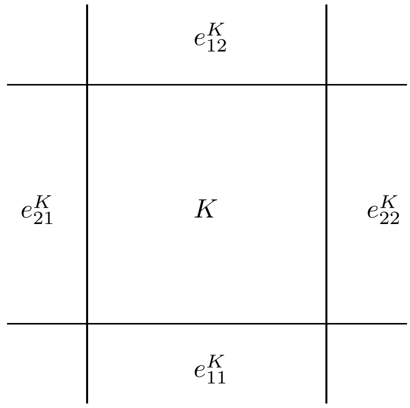

FIGURE 1. Edges e_{ij}^K of a square K

Theorem 2.1. *Assume the solutions* $u \in \mathcal{C}^3$, $\varphi \in \mathcal{C}^3$ *of* (1) *and* (3), *respectively, are for some* $\gamma \in (0,1)$ *approximated uniformly with error*

$$\begin{aligned} \|u - u_h\|_{W^{1,\infty}} + \|\varphi - \varphi_h\|_{W^{1,\infty}} &= \mathcal{O}(h_{max}^{\gamma}) \\ \|u - u_h\|_{L^\infty} + \|\varphi - \varphi_h\|_{L^\infty} &= \mathcal{O}(h_{max}^{2\gamma}), \end{aligned} \tag{15}$$

using piecewise bilinear elements for a structured mesh with at most one hanging node per edge. Then the global error (4) *has the error expansion*

$$(u - u_h, F) = \sum_{K \in \mathcal{T}} \underbrace{\left(\overline{\rho}_K + \mathcal{O}(h_{max}^{\gamma}/\alpha + \alpha)\right)}_{\equiv \check{\rho}_K} h_K^4, \tag{16}$$

with uniformly convergent computable leading order error density $\bar{\rho}$ *defined by* (14) *and* (8)-(13) *for* $\alpha^{-1} = o(h_{max}^{-\gamma})$, *satisfying*

$$\bar{\rho} = \frac{a}{12}\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 \varphi}{\partial x_2^2}\right) + \mathcal{O}(h_{max}^{\gamma}/\alpha + \alpha). \tag{17}$$

The proof of the theorem is based on the uniform convergence of the averaged second differences $\overline{D^2 u_h}$ and $\overline{D^2 \varphi_h}$, derived in Lemma 2.5, and the convergence of $h^{-2}(\pi\varphi - \varphi)$ established in Lemma 2.6. The pointwise convergence of the averaged differences is essentially a consequence of the observation that the difference operator $\bar{h}D^2$ is symmetric, which is proved in Lemma 2.4, and weak convergence. We first prove the lemmas and then the theorem.

Finite element approximations of the coercive linear problems (1) and (3), with piecewise bilinear elements, satisfy the estimate

$$\begin{aligned} \|u - u_h\|_{L^\infty} + \|\varphi - \varphi_h\|_{L^\infty} &= \mathcal{O}(h_{max}^2 \log h_{max}^{-1}), \\ \|u - u_h\|_{W^{1,\infty}} &= \mathcal{O}(h_{max}), \end{aligned} \tag{18}$$

provided $u, \varphi \in \mathcal{C}^2(\Omega)$, see [10], [12]. This estimate and Theorem 2.1 imply that the choice

$$\alpha^{-1} = o((h_{max}\sqrt{\log h_{max}^{-1}})^{-1}), \tag{19}$$

yields convergent error densities.

Let $\widetilde{\rho}$ be the limit error density in (17). Theorem 2.1 can also be used in some interesting cases when the optimal adaptive isotropic mesh uses a number of elements proportional to $(\|\widetilde{\rho}\|_{L^{\frac{d}{d+2}}}/\mathrm{TOL})^{d/2} < \infty$ while the same error for approximation with a uniform mesh requires a much larger number of elements proportional to a higher power of TOL^{-1} because $\|\widetilde{\rho}\|_{L^1} = \infty$. The assumption $u,\ \varphi \in \mathcal{C}^3$ is then violated. But for some problems it is possible to determine a TOL dependent regularization of the problems for u and φ which makes Theorem 2.1 applicable, provided the regularization is small enough for the change in the functional $(\cdot, F)$ to be less than TOL and large enough so that a refined analysis of the regularization dependent remainder in (16) makes $\mathcal{O}(h_{max}^{\gamma}/\alpha + \alpha)$ negligible, see Remark 3.6 for an example.

Remark 2.2 (Averages are needed)**.** *The uniform convergence proved in Lemma 2.5 applies to the averaged second differences $\overline{D^2 u_h}$ and $\overline{D^2\varphi_h}$. On the other hand, with uniform meshes the difference quotients without averaging converge uniformly to the corresponding second derivatives. The following example explains why the second order difference quotients of the interpolant on meshes with hanging nodes do not converge uniformly on Ω; numerical tests show that the corresponding finite element solution of Poisson's equation behaves similar to the interpolant, i.e. second order difference quotients of the finite element solution with hanging nodes do not converge uniformly. Let $u(x_1, x_2) = x_2^2$, so that $\partial_{x_1x_1}u = 0$ everywhere, and compute the second difference $D_1^2\pi u$ in the node $(-h, 0)$ neighboring the hanging node $(0, 0)$ as in Figure 2. Using bilinear finite elements the nodal interpolant πu is equal to u in all proper nodes, but in the hanging node $\pi u(0, 0) = h^2 \neq 0 = u(0, 0)$. Then $D_1^2\pi u(-h, 0) = h^2/h^2 = 1$ but $\partial_{x_1x_1}u(-h, 0) = 0$. In spite of this, Lemma 2.5 shows that the averaged difference quotients converge uniformly under the condition that $\alpha^{-1} = o(h_{max}^{-\gamma})$ as $h_{max} \to 0$.*

Remark 2.3 (Localization)**.** *One would like to take $\alpha = \alpha(x_j) \simeq h(x_j)$, so that the average is essentially based only on a few neighboring elements. Our convergence proof determines α by (18)-(19), where in particular α depends on the global pointwise error $\varphi - \varphi_h$. Remark 3.6 shows a variant of a more refined estimate where α varies in space.*

Note however that, the estimate $|\varphi(x) - \varphi_h(x)| = \mathcal{O}(h^2(x))$ does not hold for general meshes: the error representation (4) applied to the equation for φ and the dual corresponding to the pointwise error provides a function G such that $\varphi(x) - \varphi_h(x) = \int_\Omega G(x, y)h(y)^2 dy$, which implies that $|\varphi(x) - \varphi_h(x)|/h(x)^2$ may be unbounded for optimally refined meshes where $\widetilde{\rho}h^{2+d}$ is constant.

On the other hand, in practice the algorithm seems to work reasonably well even without the localized averages, cf. [8].

On a uniform mesh, the difference operator D^2 is symmetric so that summation by parts behaves like integration by parts. On an adapted non uniform mesh, the difference operator D^2 is not symmetric, however we have

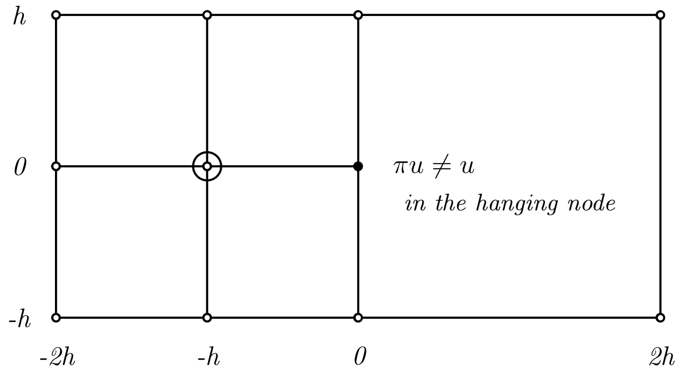

FIGURE 2. Difference quotients without averaging do not converge uniformly to second derivatives in the presence of hanging nodes.

Lemma 2.4. *The difference operator $\bar{h}D^2$ is symmetric, i.e. the diagonal matrix $\bar{h}$ is a symmetrizer for D^2, and hence, for all $v, w \in \mathbb{R}^{\bar{N}+2}$, the summation by parts formula*

$$\sum_{i=1}^{\bar{N}} \bar{h}_i D^2 v_i w_i = \sum_{i=1}^{\bar{N}} v_i \bar{h}_i D^2 w_i + \frac{v_{\bar{N}+1} w_{\bar{N}}}{h_{\bar{N}}} + \frac{v_0 w_1}{h_0} - \frac{v_{\bar{N}} w_{\bar{N}+1}}{h_{\bar{N}}} - \frac{v_1 w_0}{h_0}.$$

holds.

Proof. Summation by parts proves the lemma

$$\begin{aligned}
\sum_{i=1}^{\bar{N}} \bar{h}_i D^2 v_i w_i &= \sum_{i=1}^{\bar{N}} \left(\frac{v_{i+1} - v_i}{h_i} - \frac{v_i - v_{i-1}}{h_{i-1}}\right) w_i \\
&= \sum_{i=1}^{\bar{N}} v_i \left(\frac{w_{i+1} - w_i}{h_i} - \frac{w_i - w_{i-1}}{h_{i-1}}\right) \\
&\quad + \frac{v_{\bar{N}+1} w_{\bar{N}}}{h_{\bar{N}}} + \frac{v_0 w_1}{h_0} - \frac{v_{\bar{N}} w_{\bar{N}+1}}{h_{\bar{N}}} - \frac{v_1 w_0}{h_0} \\
&= \sum_{i=1}^{\bar{N}} v_i \bar{h} D^2 w_i \\
&\quad + \frac{v_{\bar{N}+1} w_{\bar{N}}}{h_{\bar{N}}} + \frac{v_0 w_1}{h_0} - \frac{v_{\bar{N}} w_{\bar{N}+1}}{h_{\bar{N}}} - \frac{v_1 w_0}{h_0}.
\end{aligned}$$

□

This symmetry of $\bar{h}D^2$ is the essential ingredient to obtain convergence of the averages $\overline{D^2 u_h}$ and $\overline{D^2 \varphi_h}$.

Lemma 2.5. *Assume that $u, \varphi \in \mathcal{C}^3$ and*

$$\begin{aligned}
h_{max}^{\gamma} \|u - u_h\|_{W^{1,\infty}} &+ h_{max}^{\gamma} \|\varphi - \varphi_h\|_{W^{1,\infty}} \\
&+ \|u - u_h\|_{L^\infty} + \|\varphi - \varphi_h\|_{L^\infty} = \mathcal{O}(h_{max}^{2\gamma})
\end{aligned}$$

hold, then

$$\|\overline{D_i^2 u_h} - \partial_{x_i x_i} u\|_{L^\infty} + \|\overline{D_i^2 \varphi_h} - \partial_{x_i x_i}\varphi\|_{L^\infty} = \mathcal{O}(h_{max}^\gamma/\alpha + \alpha), \quad \text{as } h_{max} \to 0. \tag{20}$$

Proof. Let $(w, w_h) = (u, u_h)$ or $(w, w_h) = (\varphi, \varphi_h)$ and consider the average error $\overline{D_i^2(w_h - w)}$. Lemma 2.4 shows that the operator $\bar{h}D_i^2$ is symmetric. This symmetry and the assumptions (10)-(11) yield

$$\begin{aligned}
|\overline{D_i^2(w_h - w)}(x_j)| &= |\bar{h}D_i^2(w_h - w)\cdot\psi^j| \\
&= \Big|(w_h - w)\cdot\bar{h}D_i^2\psi^j \\
&\quad + \frac{(w_h - w)(x_{\bar{N}+1}) - (w_h - w)(x_{\bar{N}})}{h_{\bar{N}}}\psi^j(x_{\bar{N}+1}) \\
&\quad - \frac{(w_h - w)(x_1) - (w_h - w)(x_0)}{h_0}\psi^j(x_0)\Big| \\
&\le \|w_h - w\|_{\ell^\infty}\|\bar{h}D^2\psi^j\|_{l^1} + 2\|w_h' - w'\|_{L^\infty}\max_{x\in\{0,a\}}|\psi^j(x)| \\
&= \mathcal{O}(\frac{h_{max}^\gamma}{\alpha}) \to 0,
\end{aligned}$$

provided $\alpha^{-1} = o(h_{max}^{-\gamma})$ as $h_{max} \to 0$.

The function ψ^j also satisfies the weak convergence (12) which together with the uniform convergence $D_i^2 w - \partial_{x_i x_i} w = \mathcal{O}(h_{max})$ and $w \in \mathcal{C}^3$ imply $\overline{D_i^2 w} - \partial_{x_i x_i} w = \mathcal{O}(\alpha)$. Therefore we conclude that $\overline{D_i^2 w_h} - \partial_{x_i x_i} w = \mathcal{O}(h_{max}^\gamma/\alpha + \alpha)$, uniformly as $h_{max} \to 0$. □

Lemma 2.6. *Assume that $w \in \mathcal{C}^3(\Omega)$. Consider a point $x' \in \Omega$ and a sequence of squares K containing x'. Then*

$$h_K^{-4}\int_K (\pi w(x) - w(x))\,dx - \frac{1}{12}\triangle w(x') = \mathcal{O}(h_K). \tag{21}$$

Proof. Apply the tensor property $\pi = \pi_1\pi_2$, where π_i is the nodal interpolant in the x_i direction, to split

$$\pi w - w = \pi_1(\pi_2 w - w) + \pi_1 w - w. \tag{22}$$

Translate the square K to the reference square $K_0 \equiv [0,h]\times[0,h]$, for $h = h_K$. Approximation of w by a quadratic function on K_0 shows that

$$\pi_i w - w = \frac{1}{2}\partial_{x_i x_i} w(x') x_i(h - x_i) + \mathcal{O}(h^3), \text{ on } K_0. \tag{23}$$

The integral

$$\int_0^h x_i(h - x_i)dx_i = \frac{h^3}{6}$$

combined with (22)-(23) proves the lemma. □

Proof of Theorem 2.1. The error representation (5) divides the residual into parts supported in the interior of squares and on their edges. The assumption (15) on uniform convergence of $\|u - u_h\|_{W^{1,\infty}}$, cf. [10, 12], yields convergence of $\nabla a\cdot\nabla u_h$

which combined with Lemma 2.6 and Lemma 2.5 imply convergence of the interior part

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_K (-\nabla a\cdot\nabla u_h - f)\,(\pi\varphi-\varphi)\,dx \\
&= \sum_{K\in\mathcal{T}}\left(\overline{(-\nabla a\cdot\nabla u_h - f)}(\overline{D_1^2\varphi_h}+\overline{D_2^2\varphi_h})\frac{h_K^4}{12}+h_K^4\mathcal{O}(h_{max}^\gamma/\alpha+\alpha)\right).
\end{aligned}\tag{24}
$$

The convergence of the error density associated to the edges is more subtle and uses summation by parts to shift the jump of the discrete normal derivative $\partial u_h/\partial n$ to the more regular non discrete dual weight φ. The normal derivative $\partial u_h/\partial x_i$ is the same on the opposite edges of a square. Therefore the regularity $\varphi\in\mathcal{C}^3$ and the uniform convergence $\nabla u_h\to\nabla u$ of (15) yield

$$
\begin{aligned}
&\int_{\partial K} a\frac{\partial u_h}{\partial n}(\pi\varphi-\varphi)ds \\
&= \int_{e_{12}^K}\frac{\partial u_h}{\partial x_2}\Big|_{e_{12}}\left(a\,(\pi\varphi-\varphi)\,|_{e_{12}} - a\,(\pi\varphi-\varphi)\,|_{e_{11}}\right)dx_1 \\
&\quad+\int_{e_{22}^K}\frac{\partial u_h}{\partial x_1}\Big|_{e_{22}}(a\,(\pi\varphi-\varphi)\,|_{e_{22}} - a\,(\pi\varphi-\varphi)\,|_{e_{21}})dx_2 \\
&= \int_{e_{12}^K}\frac{\partial u}{\partial x_2}\Big|_{e_{12}}(a\,(\pi\varphi-\varphi)\,|_{e_{12}} - a\,(\pi\varphi-\varphi)\,|_{e_{11}})dx_1 \\
&\quad+\int_{e_{22}^K}\frac{\partial u}{\partial x_1}\Big|_{e_{22}}(a\,(\pi\varphi-\varphi)\,|_{e_{22}} - a\,(\pi\varphi-\varphi)\,|_{e_{21}})dx_2 + h^4\mathcal{O}(h_{max}^\gamma),
\end{aligned}
$$

where $e_{ij}\equiv e_{ij}^K$. Use summation by parts over all edges and $\pi\varphi=\varphi$ on $\partial\Omega$ to obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}}\int_{\partial K} a\frac{\partial u_h}{\partial n}(\pi\varphi-\varphi)ds \\
&= -\sum_{K\in\mathcal{T}}\left(\int_{e_{12}^K} a\Big(\frac{\partial u}{\partial x_2}\Big|_{e_{12}} - \frac{\partial u}{\partial x_2}\Big|_{e_{11}}\Big)(\pi\varphi-\varphi)\,dx_1\right. \\
&\qquad\left.+\int_{e_{22}^K} a\Big(\frac{\partial u}{\partial x_1}\Big|_{e_{22}} - \frac{\partial u}{\partial x_1}\Big|_{e_{21}}\Big)(\pi\varphi-\varphi)\,dx_2 + h^4\mathcal{O}(h_{max}^\gamma)\right).
\end{aligned}\tag{25}
$$

We have

$$
h^{-1}\left(\frac{\partial u}{\partial x_2}\Big|_{e_{12}} - \frac{\partial u}{\partial x_2}\Big|_{e_{11}}\right) = \partial_{x_2x_2}u + \mathcal{O}(h_{max}),
$$

therefore Lemmas 2.5 and 2.6 applied to u_h and φ_h imply that each term in the right hand side of (25) has the expansion

$$
\begin{aligned}
\int_{e_{12}^K} a\left(\frac{\partial u}{\partial x_2}\Big|_{e_{12}} - \frac{\partial u}{\partial x_2}\Big|_{e_{11}}\right)(\pi\varphi-\varphi)\,dx_1 &= \frac{h^4}{12}a\overline{D_2^2u_h}\;\overline{D_1^2\varphi_h} \\
&\quad+ h^4\mathcal{O}(h_{max}^\gamma/\alpha+\alpha),
\end{aligned}\tag{26}
$$

and similarly for the edges in the other direction. Note that the convergence $\mathcal{O}(h_{max}^\gamma/\alpha+\alpha)\to 0$ in (26) is uniform on Ω. Finally, split the sum over all edges in (25) into a sum over all edges of squares to get the remaining factor $\frac{1}{2}$

$$\sum_{K\in\mathcal{T}} \Big(\int_{e_{12}^K} \dots dx_1 + \int_{e_{22}^K} \dots dx_2\Big)$$
$$= \sum_{K\in\mathcal{T}} \frac{1}{2}\Big(\int_{e_{11}^K} \dots dx_1 + \int_{e_{12}^K} \dots dx_1 + \int_{e_{21}^K} \dots dx_2 + \int_{e_{22}^K} \dots dx_2\Big)$$

and use (26) to complete the proof of (16). The error estimate (17) follows from Lemmas 2.4 and 2.5. □

2.3. **Efficient Computation of the Averages.** Let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

We will use $\psi^j : \mathcal{N} \to \mathbb{R}$ determined, for $j = 1, \dots, \bar{N}$, by

$$\bar{h}_i \psi_i^j - \alpha^2 \bar{h}_i D^2 \psi_i^j = \delta_{ij}, \quad i = 1, \dots, \bar{N} \tag{27}$$

with homogeneous Neumann boundary conditions $\psi_0^j = \psi_1^j$ and $\psi_{\bar{N}}^j = \psi_{\bar{N}+1}^j$. The averages $Y_j \equiv \overline{D^2 v}(x_j) = \sum_{i=1}^N \bar{h}_i D^2 v_i \psi_i^j$ then solves the dual equation to (27) with the right hand side $\bar{h}_i D^2 v_i$. The symmetry of $\bar{h} D^2$ and the summation by parts formula in Lemma 2.4 show that Y can be efficiently computed by

$$Y_j - \alpha^2 D^2 Y_j = D^2 v_j, \quad j = 1, \dots, \bar{N}$$

with homogeneous Neumann boundary conditions, $Y_0 = Y_1,\ Y_{\bar{N}} = Y_{\bar{N}+1}$.

It remains to verify the conditions (9)-(12) for ψ. The following discrete minimum principle argument first observes that $D^2 \psi_{i^*}^j \geq 0$ at a minimum point i^* and hence (27) yields non negative ψ

$$\psi_i^j \geq \psi_{i^*}^j = \frac{\delta_{i^* j}}{\bar{h}_{i^*}} + \alpha^2 D^2 \psi_{i^*}^j \geq 0.$$

The remaining conditions can be derived from the following estimate of the weighted ℓ^1 norm $\sum_i \bar{h}_i \psi_i^j w_i$, with the weights $w_i = \cosh(\frac{x_i - x_p}{\bar{\alpha}})$ for $\bar{\alpha} \geq 2\alpha$ and $x_p \in \mathcal{N}$: summation by parts in (27) shows

$$\begin{aligned} &\sum_{i=1}^{\bar{N}} \bar{\psi}_i^j \bar{h}_i (w_i - \alpha^2 D^2 w_i) + \\ &+ \alpha^2 \psi_{\bar{N}+1}^j \frac{w_{\bar{N}+1} - w_N}{h_{\bar{N}}} - \alpha^2 \psi_0^j \frac{w_1 - w_0}{h_0} = \cosh\Big(\frac{x_j - x_p}{\bar{\alpha}}\Big). \end{aligned} \tag{28}$$

We have

$$\begin{aligned} D^2 w_i &= \bar{\alpha}^{-2} w_i + \mathcal{O}(h_{max}), \\ \frac{w_{\bar{N}+1} - w_{\bar{N}}}{h_{\bar{N}}} &= \bar{\alpha}^{-1} \sinh\Big(\frac{x_{\bar{N}} - x_p}{\bar{\alpha}}\Big) + \mathcal{O}(h_{max}), \\ \frac{w_1 - w_0}{h_0} &= \bar{\alpha}^{-1} \sinh\Big(\frac{x_0 - x_p}{\bar{\alpha}}\Big) + \mathcal{O}(h_{max}). \end{aligned}$$

Therefore all terms in the left hand side of (28) are non negative for $\bar{\alpha} \geq 2\alpha$ and provide estimates to verify the conditions (10)-(12).

Note first that the choice $w = 1$, corresponding to $\bar{\alpha} = \infty$ in (28), implies $\sum_i \bar{h}_i \psi_i^j = 1$ and hence by (27) we have $\|\alpha^2 \bar{h} D^2 \psi^j\|_{\ell^1} \leq 2$.

The estimate (28) shows first that the boundary values satisfy

$$\psi_0^j = \mathcal{O}(1/\min(x_j, \alpha)),$$
$$\psi_{\bar N+1}^j = \mathcal{O}(1/\min(a - x_j, \alpha)),$$

and with a second choice of weight function $w_i = e^{-x_i/\bar\alpha}$ and $w_i = e^{(x_i-a)/\bar\alpha}$, respectively, we have similarly

$$\psi_0^j = \mathcal{O}(1/\alpha),$$
$$\psi_{\bar N+1}^j = \mathcal{O}(1/\alpha).$$

Finally, to verify the weak convergence we estimate

$$|\sum_i \psi_i^j (v_i - v_j)\bar h_i| \le \|\psi^j w \bar h\|_{\ell^1} \|(v_\cdot - v_j) w^{-1}\|_{\ell^\infty}$$

and use (28) together with a uniform bound on the difference quotients of v to obtain

$$\|(v_\cdot - v_j)w^{-1}\|_{\ell^\infty} \le \mathcal{O}(\max_i |x_i - x_j| e^{-|x_i - x_j|/\bar\alpha}) = \mathcal{O}(\bar\alpha),$$

so that $|\sum_i \psi_i^j v_i \bar h_i - v_j| \le \mathcal{O}(\bar\alpha)$.

Remark 2.7 (One dimension)**.** *Note that in one dimension, $d = 1$, the edge part of the residual vanishes since $\pi\varphi = \varphi$ in the nodes.*

Remark 2.8 (Nonlinear problems)**.** *A non linear problem $a = a(u, x)$ and $f = f(\nabla u, u, x)$ for a non linear functional, $\int_\Omega g(u(x), x)dx$, gives a different dual problem for ψ, but the same approximation property*

$$\|u - u_h\|_{W^{1,\infty}} + \|\varphi - \varphi_h\|_{L^\infty} = \mathcal{O}(h_{max}^\gamma)$$

and the regularity $u \in \mathcal{C}^3$, $\varphi \in \mathcal{C}^3$ also yield estimates of the linearization error and imply the conclusion in the theorem.

Remark 2.9 (Alternative error densities)**.** *Let $s \in [0, 1]$ and*

$$\mathcal{R}^*(\varphi_h) = -div(a\nabla\varphi_h) - F.$$

Then

$$\begin{aligned}(u - u_h, F) &= s\,(\mathcal{R}(u_h), \pi\varphi - \varphi) \\ &\quad +(1-s)\,(\pi u - u, \mathcal{R}^*(\varphi_h))\end{aligned}$$

are alternative global error estimates for $s \in [0, 1]$, cf. [8]. The piecewise bilinear approximation on structured meshes with hanging nodes in this work shows that in fact also the local error densities are the same for all $s \in [0, 1]$ asymptotically as $h_{max} \to 0$

$$\bar\rho \to \frac{a}{12}\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 \varphi}{\partial x_2^2}\right)$$

including non linear problems $a = a(u, x), f = f(\nabla u, u, x), g = g(u, x)$ and the approximation errors $\overline{D_i^2 u_h} \to \frac{\partial^2 u}{\partial x_i^2}$ and $\overline{D_i^2 \varphi_h} \to \frac{\partial^2 \varphi}{\partial x_i^2}$ are the dominating errors in the convergence.

3. Convergence Rates for the Adaptive Mesh Algorithm

This section constructs an adaptive algorithm and analyzes its stopping, accuracy and efficiency, using the convergence of the error density in Theorem 2.1. The analysis is largely based on the similar work on ordinary differential equations in [21]. The main difference is the hanging node constraint present here.

3.1. **Adaptive Refinements and Stopping.**

Theorem 2.1 proves that the error expansion

$$(u - u_h, F) = \sum_K (\overline{\rho}_K + \mathcal{O}(\frac{h_{max}^\gamma}{\alpha} + \alpha))h_K^{2+d} \equiv \sum_K \check{\rho}_K h_K^{2+d}, \tag{29}$$

has a well defined leading order error density $\overline{\rho}$ which converges uniformly as $h_{max} \to 0+$. In the adaptive algorithm below we will use the positive approximate error density $\hat{\rho}_K$ defined by

$$\hat{\rho}|_K \equiv \hat{\rho}_K \equiv \max\left(|\overline{\rho}_K|,\ \delta\right) \tag{30}$$

where

$$\delta \equiv \mathcal{O}(\sqrt{h_{max}^\gamma/\alpha + \alpha}). \tag{31}$$

The constant $\delta > 0$ is motivated by the requirements that $h_{max} \to 0$ as $\mathrm{TOL} \to 0$ and that the bounds for the error density in (40) hold, see Lemma 3.2. Let us now motivate the optimal choice of element sizes

$$\big|\rho\big| h^{d+2} = \text{constant},$$

for hypothetical linear tensor finite element methods with no other constraint than tensor cube elements and a mesh independent error density ρ. Define first, for a mesh with elements $\{K_1, K_2, K_3, \ldots, K_N\}$, the piecewise constant error density and mesh functions $\rho|_{K_i} \equiv \rho_i \equiv \rho_{K_i}$, $\hat{\rho}|_{K_i} \equiv \hat{\rho}_i \equiv \hat{\rho}_{K_i}$ and $h|_{K_i} \equiv h_i \equiv h_{K_i}$. The number of elements that corresponds to a mesh with size h can be determined by

$$N(h) \equiv \int_\Omega \frac{dx}{h^d(x)}. \tag{32}$$

It seems hard to use the sign of the error indicator for constructing the mesh, since with only two elements the error can be zero just by chance: let $\int_0^1 f(s)ds = 0$ be the integral of a continuous function where also $f(0) = f(1) = 0$. This integral can be computed by the Euler method without error for a very particular choice of just the two elements $(0, \bar{s}), (\bar{s}, 1)$, with an interior point $\bar{s}$ satisfying $f(\bar{s}) = 0$, but any other choice of two elements gives in general very large error. Instead we choose to minimize the number of elements N in (32) under the more stringent constraint

$$\sum_{i=1}^{\bar{N}} |\rho_i| h_i^{d+2} = \int_\Omega |\rho(x)| h^2 dx = \mathrm{TOL}. \tag{33}$$

This yields, with a standard application of a Lagrange multiplier, the optimal element sizes h^* satisfying

$$|\rho|(h^*)^{d+2} = \ \textit{constant} \tag{34}$$

and

$$h^* \equiv \frac{\mathrm{TOL}^{\frac{1}{2}}}{|\rho|^{\frac{1}{d+2}}} \left(\int_\Omega |\rho(x)|^{\frac{d}{d+2}} dx \right)^{-\frac{1}{2}} . \tag{35}$$

This condition is optimal only for density functions ρ with one sign and for meshes with shape regular elements, i.e. non stretched elements. To use the sign of the density or orientation of stretched elements in an optimal way is not considered in this work.

The goal of the adaptive algorithm described below is to construct a mesh of Ω such that

$$\hat{\rho}_i h_i^{d+2} \approx \frac{\mathrm{TOL}}{N}, \quad \forall\, i = 1, \dots, N, \tag{36}$$

which is an approximation of the optimal (34). To achieve (36) let $s_1 \approx 1$ be a given constant, start with an initial mesh of size $h[1]$ and then specify iteratively a new mesh $h[k+1]$, from $h[k]$, using the following dividing strategy:

(37)

for all elements $i = 1, 2, \dots, N[k]$

  $\bar{r}_i[k] \equiv \hat{\rho}_i[k](h_i[k])^{d+2}$

  if $\bar{r}_i[k] > s_1 \dfrac{\mathrm{TOL}}{N[k]}$ **then**

    mark element i for division and recursively mark all neighbors that need division due to the hanging node constraint: at most one hanging node per edge

  endif

endfor

divide all marked elements into 2^d uniform sub elements.

With this dividing strategy, it is natural to use the stopping criterion:

$$\textbf{if} \quad \left(\max_{1 \le i \le N[k]} \bar{r}_i[k] \le S_1 \frac{\mathrm{TOL}}{N[k]} \right) \quad \textbf{then} \quad \text{stop.} \tag{38}$$

Here S_1 is a given constant, with $S_1 > s_1 \approx 1$, determined more precisely as follows: we want that the maximal error indicator decays quickly to the stopping level $S_1\mathrm{TOL}/N$, but when almost all error indicators $\bar{r}_i$ satisfy $\bar{r}_i < s_1 \frac{\mathrm{TOL}}{N}$ the reduction of the error may be slow. Theorem 3.1 shows that slow reduction is avoided if S_1 satisfies (41).

The remainder of this section analyzes in three theorems the adaptive algorithm based on (36) with respect to stopping, accuracy and efficiency. To analyze the decay of the maximal error indicator, it is useful to understand the variation of the density $\hat{\rho}$ at different refinement levels, in particular we will consider an element $K[k]$ and its parent on a previous refinement level, $p(K, k)$, with the corresponding error density $\hat{\rho}(K)[p(K, k)]$. Since by (47), TOL $\to 0+$ implies that $h_{max} \to 0$, Theorem 2.1 shows that there is a limit error density $\widetilde{\rho}$ such that

$$\check{\rho} \to \widetilde{\rho},\ \overline{\rho} \to \widetilde{\rho} \text{ and } \hat{\rho} \to |\widetilde{\rho}|, \text{ as } \mathrm{TOL} \to 0+ . \tag{39}$$

A consequence of $\hat{\rho} \to |\widetilde{\rho}|$, as TOL $\to 0+$, and (30) is that for all elements K and all refinement levels k there exists a positive function c close to 1 for sufficiently refined meshes, such that the error density satisfies

$$\begin{aligned} c(K) &\le \frac{\hat\rho(K)[p(K,k)]}{\hat\rho(K)[k]} \le c(K)^{-1}, \\ c(K) &\le \frac{\hat\rho(K)[k-1]}{\hat\rho(K)[k]} \le c(K)^{-1}, \end{aligned} \tag{40}$$

provided $\max_{K,k} h_K[k]$ is sufficiently small. In other words, (40) holds with e.g. $c = 2^{-1}$ for sufficiently small $\max_{K,k} h_K[k]$. Note that the condition (40) also implies a related constraint on the optimal mesh, see Remark 3.3.

Theorem 3.1 (Stopping)**.** *Suppose the assumptions of Theorem 2.1 hold and the adaptive algorithm uses the strategy* (37)-(38)*. Assume that c satisfies* (40)*, for the elements corresponding to the maximal error indicator on each refinement level, and that*

$$S_1 \ge \frac{2^d}{c} s_1, \quad 1 > \frac{c^{-1}}{2^{d+2}}. \tag{41}$$

Then each refinement level either decreases the maximal error indicator with the factor

$$\max_{1\le i\le N[k+1]} \bar r_i[k+1] \le \frac{c^{-1}}{2^{d+2}} \max_{1\le i\le N[k]} \bar r_i[k], \tag{42}$$

or stops the algorithm.

We have tested several alternative stopping rules, such as $|\sum_i \bar\rho_i h_i^{2+d}| \le \mathrm{TOL}$ which may be the first idea for a stopping condition. It turns out that the stopping condition (38) yields more accurate error estimates both theoretically and computationally, see [21].

Proof. Define the piecewise constant error indicator function $\bar r|_K \equiv \bar r_K$, for all elements K. There is a point $x^* \in \Omega$ giving the maximal error indicator value

$$\bar r(x^*)[k+1] = \max_{1\le i\le N[k+1]} \bar r_i[k+1]$$

on refinement level $k+1$. The corresponding indicator $\bar r(x^*)[k]$, on the previous level, satisfies precisely one of the following three statements

$$\bar r(x^*)[k] \le \frac{s_1 \mathrm{TOL}}{N[k]}, \tag{43}$$

$$\frac{s_1 \mathrm{TOL}}{N[k]} < \bar r(x^*)[k] \le 2^{d+2}\frac{s_1 \mathrm{TOL}}{N[k]}, \tag{44}$$

$$\bar r(x^*)[k] > 2^{d+2}\frac{s_1 \mathrm{TOL}}{N[k]}. \tag{45}$$

If (43) holds either the element containing x^* is not divided on level $k+1$ or it is divided on level $k+1$ by the hanging node condition. In any case, (40) implies

$$\bar r(x^*)[k+1] \le \frac{c^{-1} s_1 \mathrm{TOL}}{N[k]}. \tag{46}$$

Condition (41) and the bound $N[k+1] \le 2^d N[k]$ imply

$$\frac{S_1 \mathrm{TOL}}{N[k+1]} \ge \frac{c^{-1} s_1 \mathrm{TOL}}{N[k]},$$

which together with (46) show that the algorithm stops at level $k+1$ if (43) holds.

Similarly, if (44) holds, the element containing x^* is divided on level $k+1$, so that $\bar{r}(x^*)[k+1] \le \frac{c^{-1}s_1\text{TOL}}{N[k]}$ again and consequently the algorithm stops at level $k+1$.

Finally if (45) holds, the elements containing x^* is divided and by (40)

$$\bar{r}(x^*)[k+1] \le \frac{c^{-1}}{2^{d+2}}\bar{r}(x^*)[k] \le \frac{c^{-1}}{2^{d+2}} \max_{1\le i\le N[k]} \bar{r}_i[k],$$

which proves the theorem. □

Let us verify that the choice (31) of δ implies that $h_{max} \to 0$ and that the function c in (40) is close to 1 for sufficiently refined meshes.

Lemma 3.2. *Suppose the assumptions of Theorem 3.1 hold, then*

$$\lim_{\text{TOL}\to 0+} h_{max}[J] = 0, \tag{47}$$

for the final mesh J, and

$$\left|\frac{\hat{\rho}(K)[p(K,k)]}{\hat{\rho}(K)[k]} - 1\right| = \mathcal{O}(\sqrt{h_{max}^{\gamma}/\alpha + \alpha}),$$
$$\left|\frac{\hat{\rho}(K)[k-1]}{\hat{\rho}(K)[k]} - 1\right| = \mathcal{O}(\sqrt{h_{max}^{\gamma}/\alpha + \alpha}),$$

where $h_{max} \equiv \max_{k\le J} h_{max}[k]$.

Proof. When the algorithm stops, on level J, the error indicators satisfy the bound

$$\hat{\rho}_i h_i^{d+2} \le \frac{S_1\text{TOL}}{N}, \quad \text{for all } i. \tag{48}$$

Consequently we have by (30)

$$\frac{S_1\text{TOL}}{N} \ge \delta h_{max}^{d+2},$$

which using (31) proves (47):

$$h_{max}^2\sqrt{h_{max}^{\gamma}/\alpha + \alpha} \le \frac{S_1\text{TOL}}{N h_{max}^d} \le \frac{S_1\text{TOL}}{\int_\Omega dx}.$$

Theorem 2.1 and definition (30) imply

$$\hat{\rho} = \max(|\widetilde{\rho}| + \mathcal{O}(h_{max}^{\gamma}/\alpha + \alpha), \delta)$$

where $|\widetilde{\rho}|$ is the limit of $\hat{\rho}$, formulated in the motivation of definition (40). Therefore, we have

$$\left|\frac{\hat{\rho}(K)[p(K,k)]}{\hat{\rho}(K)[k]} - 1\right| \le \frac{\mathcal{O}(h_{max}^{\gamma}/\alpha + \alpha)}{\delta} = \mathcal{O}(\sqrt{h_{max}^{\gamma}/\alpha + \alpha}).$$

The same estimates for $\frac{\hat{\rho}(K)[k-1]}{\hat{\rho}(K)[k]}$ finishes the proof. □

Remark 3.3 (Mesh constraints)**.** *The error density condition* (40) *also implies constraints on the optimal mesh, for instance the assumption* $\frac{1}{2}(\bar{\rho}(x)[k] + \bar{\rho}(x+h)[k]) = \bar{\rho}(x)[k-1]$ *shows that*

$$2c - 1 \le \left|\frac{\bar{\rho}(x+h)[k]}{\bar{\rho}(x)[k]}\right| \le 2c^{-1} - 1. \tag{49}$$

3.2. **Accuracy of the Adaptive Algorithm.** The adaptive algorithm guarantees that the estimate of the global error is bounded by a given error tolerance, TOL. An important question is whether the true global error is bounded by TOL asymptotically. Using the upper bound (38) of the error indicators and the convergence of ρ and $\bar\rho$ in Theorem 2.1, the global error has the estimate

Theorem 3.4 (Accuracy)**.** *Suppose that the assumptions of Theorem 3.1 hold. Then the adaptive algorithm* (37)-(38) *satisfies*

$$\limsup_{\mathrm{TOL}\to 0+} \Big(\mathrm{TOL}^{-1}\big|(u-u_h,F)\big| \Big) \le S_1. \tag{50}$$

Proof. When the adaptive algorithm stops, (16), (30) and (38) imply

$$\begin{aligned}
\mathrm{TOL}^{-1}|(u-u_h,F)| &= \mathrm{TOL}^{-1}|\sum_{i=1}^N(\bar\rho_i h_i^{d+2}+\mathcal{O}(\delta^2)h_i^{d+2})| \\
&\le \mathrm{TOL}^{-1}\sum_{i=1}^N(|\bar\rho_i|+\mathcal{O}(\delta^2))h_i^{d+2} \\
&\le \sum_{i=1}^N(1+\frac{\mathcal{O}(\delta^2)}{\hat\rho_i})\frac{S_1}{N} \\
&\le S_1+\mathcal{O}(\max\delta),
\end{aligned} \tag{51}$$

which together with Lemma 3.2 proves (50) in the limit as TOL $\to 0$. □

3.3. **Efficiency of the Adaptive Algorithm.** An important issue for the adaptive method is its efficiency: we want to determine a mesh with as few elements as possible providing the desired accuracy. From the definition (32) and the optimality condition (35), the number of optimal adaptive elements, N^{opt}, satisfies

$$N^{\mathrm{opt}} = \int_\Omega \frac{dx}{(h^*(x))^d} = \frac{1}{\mathrm{TOL}^{\frac d2}}\left(\int_\Omega |\rho[k](x)|^{\frac{d}{d+2}}dx\right)^{\frac{d+2}{2}},$$

i.e.

$$N^{\mathrm{opt}} = \frac{1}{\mathrm{TOL}^{\frac d2}}\|\rho\|^{\frac d2}_{L^{\frac{d}{d+2}}}. \tag{52}$$

Here $\|\cdot\|_{L^{\frac{d}{d+2}}}$ is the quasi-norm defined by

$$\|f\|_{L^{\frac{d}{d+2}}} \equiv \Big(\int_\Omega |f(x)|^{\frac{d}{d+2}}dx\Big)^{\frac{d+2}{d}}.$$

On the other hand, for the uniform mesh with elements $h = constant$, the number of elements, N^{uni}, to achieve $\sum_{i=1}^N|\rho_i|h_i^{d+2} = \mathrm{TOL}$ becomes

$$N^{\mathrm{uni}} = \int_\Omega \frac{dx}{h^d(x)} = \frac{\int_\Omega dx}{\mathrm{TOL}^{\frac d2}}\left(\int_\Omega |\rho[k](x)|dx\right)^{\frac d2},$$

i.e.

$$N^{\mathrm{uni}} = \frac{\int_\Omega dx}{\mathrm{TOL}^{\frac d2}}\|\rho\|^{\frac d2}_{L^1}. \tag{53}$$

Hence, the number of uniform elements is measured in the L^1-norm while the optimal number of elements is measured in the $L^{\frac{d}{d+2}}$ quasi-norm. Jensen's inequality implies $\|f\|_{L^{\frac{d}{d+2}}} \leq (\int_\Omega dx)^{\frac{2}{d}} \|f\|_{L^1}$, therefore an adaptive method may use fewer elements than the uniform element size method, cf. Remark 3.6.

The following theorem uses a lower bound of the error indicators, obtained from the stopping condition (38) and the ratio of the error density (40), to show that the algorithm (37)-(38) generates a mesh which is optimal, up to a multiplicative constant.

Theorem 3.5 (Efficiency). *Assume that $c = c(x)$ satisfies* (40) *for all elements at the final refinement level, that all initial elements have been divided when the algorithm stops and that the assumptions of Theorem 3.1 hold. Then there exists a constant $C > 0$, bounded by $(\frac{2^{d+2}}{s_1})^{\frac{d}{2}}$, such that the final number of adaptive elements N, of the algorithm* (37)-(38)*, satisfies*

$$(\mathrm{TOL}^{\frac{d}{2}} N) \leq C \, \|\frac{\hat\rho}{c}\|^{\frac{d}{2}}_{L^{\frac{d}{d+2}}} \leq C \left(\max_{x\in\Omega} c(x)^{-\frac{d}{2}} \right) \|\hat\rho\|^{\frac{d}{2}}_{L^{\frac{d}{d+2}}}, \tag{54}$$

and

$$\lim_{\mathrm{TOL}\to 0+} \|\hat\rho\|_{L^{\frac{d}{d+2}}} = \|\widetilde\rho\|_{L^{\frac{d}{d+2}}},$$
$$\lim_{\mathrm{TOL}\to 0+} \max_{x\in\Omega} c(x)^{-\frac{d}{2}} = 1,$$

i.e. the number of elements is asymptotically optimal up to the problem independent factor $C \leq (\frac{2^{d+2}}{s_1})^{\frac{d}{2}}$.

Proof. When the adaptive algorithm stops, on level k, the error indicators satisfy the upper bound

$$\bar r_K[k] = (\hat\rho(K) h_K^{d+2})[k] \leq \frac{S_1 \mathrm{TOL}}{N[k]}, \quad \forall K \in \mathcal{T}.$$

By assumption, each element $K[k]$ has a parent on a previous level, $p(K,k)$ (not necessary the previous level $k-1$), which was divided. We shall show that this parent indicator $\bar r(K)[p(K,k)]$ satisfies the lower bound

$$\bar r(K)[p(K,k)] > \frac{s_1 \mathrm{TOL}}{N[p(K,k)]}, \tag{55}$$

and this lower bound is the essential step to obtain the estimate (54). If this parent was not refined by hanging node constraints, the lower bound holds. In fact, it also holds if the refinement was made by hanging node constraints: then the parent has a refined neighbor element which has half the mesh size while the error densities $\hat\rho_i$ and $\hat\rho_j$ of two neighboring elements satisfy by Theorem 2.1

$$\begin{aligned} \big|1 - \frac{\hat\rho_j}{\hat\rho_i}\big| &= \frac{|\hat\rho_i - \widetilde\rho_i + \widetilde\rho_i - \widetilde\rho_j + \widetilde\rho_j - \hat\rho_j|}{\hat\rho_i} \\ &= \mathcal{O}(\frac{h_{max}^\gamma/\alpha + \alpha}{\delta} + \frac{h_{max}}{\delta}) = \mathcal{O}(\sqrt{h_{max}^\gamma/\alpha + \alpha}). \end{aligned}$$

Therefore the error indicator of the parent is a factor 2^{d+1} larger than for the neighbor. Hence, starting from source elements, where the indicator is marked for refinement not by the hanging node constraint, the error density for successive connected hanging node neighbors increase and consequently also the hanging node refinements satisfy the lower bound (55).

The indicators of the parent elements therefore satisfy the lower bound

$$\begin{aligned}\hat\rho(K)[p(K,k)]2^{d+2}h_K^{d+2}[k] &= (\hat\rho(K)h^{d+2}(K))[p(K,k)]\\ &> \frac{s_1\mathrm{TOL}}{N[p(K,k)]}\\ &\geq \frac{s_1\mathrm{TOL}}{N[k]}.\end{aligned}$$

The estimate on the number of elements now follows by relating the error indicators to the lower bounds of their parents:

$$\begin{aligned}h^{d+2}(K)[k] &> \frac{s_1\mathrm{TOL}}{N[k]}\frac{1}{2^{d+2}}\frac{1}{\hat\rho(K)[p(K,k)]}\\ &\geq \frac{s_1\mathrm{TOL}}{N[k]2^{d+2}}\frac{c}{\hat\rho(K)[k]}.\end{aligned}$$

This and (32) imply

$$N[k] = \int_\Omega \frac{dx}{h^d(x)[k]} < \frac{(N[k])^{\frac{d}{d+2}}2^d}{(s_1\mathrm{TOL})^{\frac{d}{d+2}}}\int_\Omega \left|\frac{\hat\rho}{c}\right|^{\frac{d}{d+2}}dx$$

which together with Hölder's inequality proves the theorem

$$\begin{aligned}N[k] &\leq \left(\frac{2^{d+2}}{s_1}\right)^{\frac{d}{2}}\left(\frac{1}{\mathrm{TOL}}\right)^{\frac{d}{2}}\|\frac{\hat\rho}{c}\|^{\frac{d}{2}}_{L^{\frac{d}{d+2}}}\\ &\leq \left(\frac{2^{d+2}}{s_1}\right)^{\frac{d}{2}}\left(\frac{1}{\mathrm{TOL}}\right)^{\frac{d}{2}}(\|c^{-1}\|_{L^\infty}\|\hat\rho\|_{L^{\frac{d}{d+2}}})^{\frac{d}{2}}.\end{aligned}$$

□

Remark 3.6 (Adaptive gain)**.** *An example with adaptive gain is Poisson's equation*

$$\begin{aligned}-\Delta u &= f \quad in\ \Omega = (-1,1)^2,\\ u &= 0 \quad on\ \partial\Omega,\end{aligned}$$

with computation of the functional

$$(u,F) = \int_\Omega \partial_{x_1}u(x)\exp\left(-|x|^2/\epsilon^2\right)dx/(\pi\epsilon^2), \tag{56}$$

and $\epsilon > 0$. Smooth f implies that $u,\varphi \in \mathcal{C}^3(\Omega)$ and that the error density has the bound

$$\rho(x) = \frac{\mathcal{O}(1)}{(|x|+\epsilon)^3},$$

which by (54) implies that the optimal number of adaptive elements N, for error TOL*, satisfies*

$$\lim_{\mathrm{TOL}\to 0+}(\mathrm{TOL}\ N) = \mathcal{O}(1),$$

while (53) shows that the number of uniform elements for the same error becomes larger by a factor ϵ^{-1}

$$N^{uni} = \frac{\mathcal{O}(\epsilon^{-1})}{\mathrm{TOL}}. \tag{57}$$

The other case with the functional $(u,F) = \partial_{x_1}u(0)$ yields an error density with a singularity. Adaptivity for this functional can also be analyzed and computed by *regularizing with* $\epsilon = o(\text{TOL})$ *in* (56) *and choosing* $\alpha = o(\text{TOL}^{1/6})|x|^{11/12}$. *Then the error density in* (16) *has the expansion*

$$\check{\rho} = \frac{\mathcal{O}(1)}{(|x| + \epsilon)^3}(1 + o(1)),$$

as TOL $\to 0+$,*and the number of adaptive elements has the asymptotic optimal bound* $\lim_{\text{TOL}\to 0+}(\text{TOL}\ N) = \mathcal{O}(1)$, *while the number of uniform elements satisfies* (57).

3.4. **Implementation of the Adaptive Algorithm.** This subsection presents a detailed implementation, called `MSST`, of the adaptive algorithm (37)-(38). The dividing strategy (37) is applied iteratively until the approximate solution is sufficiently resolved, in other words, until the approximate error density $\hat{\rho}$ and the elements satisfy the stopping criteria (38):

Initialization: The user chooses

(1) an initial error tolerance, TOL,
(2) an initial coarse (uniform) mesh and
(3) a number, s_1, in (37) and a rough estimate of c in (40) to compute S_1 using (41).

Set the iteration number k to 0.

Step I: Increase the iteration number k by 1. Compute the second order accurate approximation $u_h[k] \in V_h[k]$ of (2) and compute the approximate weight $\varphi_h[k] \in V_h[k]$, using the second order accurate method (7).

Step II: **If:** $\big(\max_{1\le i\le N[k]} \bar{r}_i[k] \le \frac{S_1 \text{TOL}}{N[k]}\big)$ **then** stop the program
 else: do (37)
  go to Step I.
 endif:

3.5. **Decreasing Tolerance.** This subsection studies an adaptive algorithm allowing the tolerance to decrease slightly as the mesh is refined. The decreasing tolerance is motivated by efficiency − the efficiency of the algorithm depends on the total work including all refinement levels. If the number of elements in each refinement iteration increases only very slowly, the total work becomes proportional to the product of the number of elements in the finest mesh times the number of refinement levels. The condition (35) shows that the number of refined levels, J, satisfies

$$\min h = 2^{-J} \int_\Omega dx/N[1] = \mathcal{O}(\text{TOL}^{1/2}). \tag{58}$$

A relation $\min h = \mathcal{O}(\text{TOL}^\alpha)$, $\alpha > 0$ still holds for many singular densities, as in Remark 3.6. Therefore, $J = \mathcal{O}(\frac{d}{2}\log(\text{TOL}^{-1})) \simeq \log N$, so that the total number of elements for the algorithm (37)-(38) would be essentially bounded by

$$\mathcal{O}(N \log N). \tag{59}$$

A more efficient refinement algorithm is obtained by successively decreasing the tolerance, $\text{TOL}[k+1] < \text{TOL}[k]$, in each refinement so that

$$\frac{N[k]}{N[k+1]} \le \bar{c} < 1 \tag{60}$$

always holds. The condition (60) would imply that the total number of elements satisfy

$$\sum_{k=1}^{J} N[k] \le \frac{N[J]}{1-\bar{c}}. \tag{61}$$

Therefore, a slightly decreasing tolerance may be more efficient than a constant tolerance, which yields the total work (59). Including the assumption

$$c' \le \frac{\mathrm{TOL}[k+1]}{\mathrm{TOL}[k]} \le 1 \tag{62}$$

and replacing c by $c'c$ in (41) directly generalizes Theorems 3.1, 3.4 and 3.5 to slightly varying tolerance, where TOL in (50) and (54) then denotes the final stopping tolerance. However, an unattractive consequence of varying tolerance is that the stopping tolerance becomes a priori uncertain, see Remark 3.7 and Theorem 3.8.

Remark 3.7. *A decreasing tolerance is useful if there are few elements with their error indicators, $\bar{r}_i$, in the set $(s_1\mathrm{TOL}/N, \infty)$. To include a decreasing tolerance, modify the algorithm by adding the command "Set $\mathtt{V} = 0$" in the end of* **Step I** *and replace the statement "* **go to Step I**" *before* **goto Step I** *in the end of* **Step II** *by:*

if: $(N[k]/N[k+1] > \bar{c}$ & $\mathtt{V} = 0)$, **then**
$\mathrm{TOL} \equiv \mathrm{TOL}[k](1 - \frac{\bar{c}^{-1}-1}{2^d-1})$, $\mathtt{V} = 1$ *and* **go to Step II**,
else:
go to Step I.
endif:

Include in the initialization also a choice of the factor $\bar{c}$ to increase the number of elements in (60)*.*

Assume that the set $(c's_1\mathrm{TOL}/N, s_1\mathrm{TOL}/N]$ contains a fraction $c''N$ of the elements, where $2^{-d} < c' < 1$; for instance, if the error indicators, $\bar{r}_i$, are uniformly distributed in $[0, s_1\mathrm{TOL}/N]$, with a negligible part outside of this set, there holds $c'' = 1 - c'$, which yields $\bar{c} = \frac{1}{1+c''(2^d-1)} = \frac{1}{1+(1-c')(2^d-1)}$ and motivates $c' = 1 - \frac{\bar{c}^{-1}-1}{2^d-1}$ in the algorithm. A refinement approximately maps the error indicator set

$$(c's_1\mathrm{TOL}/N, s_1\mathrm{TOL}/N]$$

to

$$(c's_1\mathrm{TOL}/(N2^{d+2}), s_1\mathrm{TOL}/(N2^{d+2})].$$

Then the next refinement continues with essentially a similar distribution of the error indicators, provided c' is not too small. When the algorithm stops, the final tolerance satisfies

$$\mathrm{TOL}[0] \ge \mathrm{TOL}[J] \ge \mathrm{TOL}[0](c')^J = \mathrm{TOL}^{1+\mathcal{O}(|\log c'|)},$$

which for c' close to 1 is only a slight change. □

Let us now show that the total number of elements can be bounded by a constant times the number of elements in the finest mesh, in the case of decreasing tolerance. Its proof uses that the tolerance decreases sufficiently, which simplifies the analysis. A more refined study, with less demanding assumptions on the tolerance, following the idea in Remark 3.7 would need deeper understanding of the distribution of the error indicators $\bar{r}_i$. In contrast to the basic Theorems 3.1, 3.4 and 3.5, the following result has the drawback that it uses a uniform bound in (40) which yields a condition, on c', that in practice can be too restrictive although it seems reasonable for very small tolerances. The proof is also more complicated and less natural than the previous proofs.

Theorem 3.8. *The total number of elements satisfies the bound*

$$\sum_{k=1}^{J} N[k] = \mathcal{O}(N[J]),$$

for a variant of `MSST` *where all levels have decreasing tolerance*

$$\mathrm{TOL}[k+1] = \mathrm{TOL}[k]c'$$

satisfying $0 < c' < c$, *provided all initial elements are divided,* $S_1 \geq s_1 2^d/(cc')$ *and* (40) *holds uniformly for all elements.*

Proof. Let $s_2 \equiv s_1 c/(c' 2^{d+2})$ and $\mathcal{N}_0[k] \equiv \{i : s_2 \mathrm{TOL}[k]/N[k] \leq \bar{r}_i[k] \leq s_1 \mathrm{TOL}[k]/N[k]\}$. We shall first verify that

$$\min_{K,k} (\bar{r}_K N/\mathrm{TOL})[k] \geq s_2.$$

Assume first that

$$\min_K \bar{r}_K[k] > s_1 \mathrm{TOL}/N[k],$$

then all elements are divided on level $k+1$ and by (40)

$$\begin{aligned}
\bar{r}(K)[k+1] &= (\hat{\rho}(K) h(K)^{d+2})[k+1] \\
&\geq c\hat{\rho}(K)[k] \frac{h^{d+2}(K)[k]}{2^{d+2}} \\
&= \frac{c}{2^{d+2}} \bar{r}(K)[k] \\
&> \frac{c s_1 \mathrm{TOL}[k]}{2^{d+2} N[k]}
\end{aligned}$$

therefore

$$\begin{aligned}
\min_K \bar{r}(K)[k+1] &> \frac{c s_1 \mathrm{TOL}[k+1]}{c' 2^{d+2} N[k+1]} \\
&= \frac{s_2 \mathrm{TOL}[k+1]}{N[k+1]}.
\end{aligned}$$

Then if $n \in \mathcal{N}_0[k]$ the element n is not divided on level $k+1$, unless the hanging node constraint required division but then the error indicator is bigger than its source of the hanging node constraint, see the proof of Theorem 3.5, so that

$$\begin{aligned}\bar{r}(K)[k+1] &\geq c\, \bar{r}(K)[k]\\ &\geq c\, s_2\ \mathrm{TOL}[k]/N[k]\\ &\geq \frac{c}{c'}\, s_2\ \mathrm{TOL}[k+1]/N[k+1]\\ &> s_2\ \mathrm{TOL}[k+1]/N[k+1].\end{aligned}$$

Therefore we conclude, by induction, that the error indicators satisfy

$$\min_{K,k} (\bar{r}(K)N/\mathrm{TOL})[k] \geq s_2.$$

The next step is show that at most m consecutive levels can have the slow increase $N[k]/N[k+1] > \bar{c}$. This will imply that the total number of elements is bounded by a constant times the final number of elements. Assume the contrary that

$$\frac{N[k]}{N[k+1]} > \bar{c}, \quad k = \kappa, \dots, \kappa + m, \tag{63}$$

where m and $\bar{c}$ satisfy

$$\frac{(c')^m}{c} < \frac{s_2}{s_1}, \tag{64}$$

$$1 < \bar{c}^{-1} < 2^{d/(m+1)}, \tag{65}$$

and let $N_0[k] \equiv \#\mathcal{N}_0[k]$ and $N_+ \equiv N - N_0$. The condition (63) and

$$N[k+1] = N_0[k] + 2^d N_+[k]$$

show that the number of divided elements, $N_+[k]$, satisfies

$$N_+[k] < \frac{\bar{c}^{-1} - 1}{2^d - 1} N[k]. \tag{66}$$

The tolerance decreases, so that after m levels the dividing barrier is

$$s_1 \mathrm{TOL}[\kappa + m]/N[\kappa + m] < (c')^m s_1 \mathrm{TOL}[\kappa]/N[\kappa].$$

All elements in $\mathcal{N}_0[\kappa]$ must have been divided after m levels, since if they have not all been divided some error indicators are larger than $cs_2\mathrm{TOL}[\kappa]/N[\kappa]$ and condition (64) gives the contradiction

$$s_1 \frac{\mathrm{TOL}[\kappa + m]}{N[\kappa + m]} < (c')^m s_1 \frac{\mathrm{TOL}[\kappa]}{N[\kappa]} < cs_2 \frac{\mathrm{TOL}[\kappa]}{N[\kappa]}.$$

Dividing of all elements in $\mathcal{N}_0[\kappa]$ shows that $N_0[\kappa]$ must be smaller than the sum of divided elements

$$N_0[\kappa] \leq \sum_{j=1}^{m} N_+[\kappa + j] \tag{67}$$

which also leads to a contradiction, since by (66)

$$N_0[\kappa] = N[\kappa] - N_+[\kappa] > \frac{2^d - \bar{c}^{-1}}{2^d - 1} N[\kappa]$$

and by combining (66) and (63)

$$\begin{aligned} N_+[\kappa+j] &< \frac{\bar{c}^{-1}-1}{2^d-1} N[\kappa+j] \\ &< \frac{\bar{c}^{-1}-1}{2^d-1} \bar{c}^{-1} N[\kappa+j-1] \\ &< \frac{\bar{c}^{-1}-1}{2^d-1} \bar{c}^{-j} N[\kappa], \end{aligned}$$

so that by the assumption (65)

$$N_0 - \sum_{j=1}^{m} N_+[\kappa+j] > \frac{2^d - \bar{c}^{-m-1}}{2^d-1} N[\kappa] > 0,$$

which contradicts (67). Hence, the number of consecutive levels, where $N[k]/N[k+1] > \bar{c}$, must be smaller than $m+1$ and therefore

$$\sum_{k=1}^{J} N[k] \leq \frac{mN[J]}{1-\bar{c}} = \mathcal{O}(N[J]).$$

□

References

[1] M. Ainsworth and J. T. Oden, A posteriori error estimation in finite element analysis, Comput. Methods Appl. Mech. Engrg., **142** (1997), 1-88.

[2] I. Babuška, A. Miller and M. Vogelius, Adaptive methods and error estimation for elliptic problems of structural mechanics, in *Adaptive computational methods for partial differential equations* (SIAM, Philadelphia, Pa., 1983) 57-73.

[3] I. Babuska and W. C. Rheinboldt, Error estimates for adaptive finite element computations, SIAM J. Numer. Anal., **15**(1978), 736–754.

[4] I. Babuska, T. Strouboulis and S. K. Gangaraj, Guaranteed computable bounds for the exact error in the finite element solution. I. One dimensional model problem, Comput. Methods Appl. Mech. Engrg., **176** (1999), 51–79.

[5] I. Babuška, and M. Vogelius, Feedback and adaptive finite element solution of one-dimensional boundary value problems, Numer. Math. **44** (1984), no. 1, 75-102.

[6] R. E. Bank and A. Weiser, *Some a posteriori error estimators for elliptic partial differential equations*, Math. Comp. **44** (1985), 283–301.

[7] R. Becker and R. Rannacher, A feed-back approach to error control in finite element methods: basic analysis and examples, East-West J. Numer. Math., **4** (1996), no. 4, 237-264.

[8] R. Becker and R. Rannacher, An optimal control approach to a posteriori error estimation in finite element methods, Acta Numerica, (2001), 1-102.

[9] P. Binev, W. Dahmen and R. DeVore, Adaptive finite element methods with convergence rates, Preprint 2002.

[10] S. C. Brenner and L. R. Scott, The mathematical theory of finite element methods, Texts in Applied Mathematics 15, Springer–Verlag, New York, 1994.

[11] A. Cohen, W. Dahmen and R. DeVore, Adaptive wavelet methods for elliptic operator equations: convergence rates, Math. Comp., **70** (2001), no. 233, 25-75

[12] F. Christian and G. Santos, A posteriori estimators for nonlinear elliptic partial differential equations, J. Comput. Appl. Math., **103** (1999), 99–114.

[13] R. A. DeVore, Nonlinear approximation, Acta Numerica, (1998), 51-150.

[14] W. Dörfler, A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal. **33** (1996), no. 3, 1106-1124.

[15] Edelsbrunner, H.; Grayson, D. R., Edgewise subdivision of a simplex. ACM Symposium on Computational Geometry (Miami, FL, 1999). Discrete Comput. Geom., **24** (2000), 707–719.

[16] K. Eriksson, D. Estep, P. Hansbo and C. Johnson, Introduction to adaptive methods for differential equations, Acta Numerica, (1995), 105-158.

[17] C. Johnson and A. Szepessy, Adaptive finite element methods for conservation laws based on a posteriori error estimates, Comm. Pure Appl. Math., **48** (1995), 199-234.

[18] L. Machiels, A. T. Patera, J. Peraire and Y. Maday, A general framework for finite element a posteriori error control: application to linear and nonlinear convection-dominated problems, Preprint (presentation at ICFD Conference on Numerical Methods for Fluid Dynamics, Oxford, England, March 31-April 3, 1998)

[19] Y. Maday, A. T. Patera and J. Peraire, A general formulation for a posteriori bounds for output functionals of partial differential equations; applications to the eigenvalue problem, C. R. Acad. Sci. Paris Sér. I Math., **328** (1999), 823–828.

[20] K.-S. Moon, *Convergence rates of adaptive algorithms for deterministic and stochastic differential equations*, (Licentiate thesis, ISBN 91-7283-196-0, Royal Institute of Technology, 2001) http://www.nada.kth.se/∼moon

[21] K.-S. Moon, A. Szepessy, R. Tempone and G.E. Zouraris, Convergence rates for adaptive approximation of ordinary differential equations, Numer. Math. (in press) DOI 10.1007/s00211-003-0466-9 (2003)

[22] K.-S. Moon, A. Szepessy, R. Tempone and G.E. Zouraris, Convergence rates for adaptive weak approximation of stochastic differential equations, Preprint 2002.

[23] K.-S. Moon, A. Szepessy, R. Tempone and G.E. Zouraris, Hyperbolic differential equations and adaptive numerics, in *Theory and numerics of differential equations* (Eds. J.F. Blowey, J.P. Coleman and A.W. Craig, Durham 2000, Springer Verlag, 2001)

[24] P. Morin, R. Nochetto and K.G. Siebert, Data oscillation and convergence of adaptive FEM, SIAM J. Numer. Anal., **38** (2000), 466-488.

[25] R. Stevenson, An optimal adaptive finite element method, Preprint 2003.

[26] A. Szepessy, R. Tempone and G. E. Zouraris, Adaptive weak approximation of stochastic differential equations, Comm. Pure Appl. Math., **54** (2001), 1169-1214.