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Convergence Rates of Adaptive Algorithms for Deterministic and Stochastic Differential Equations

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Abstract

An error representation of the form “Global error = \sum local error \cdot weight” is derived by a variational principle for general approximation of functions of solutions to ordinary differential equations. An adaptive algorithm including dividing and merging of time steps, based on the error expansion, is proven to stop with the optimal number of steps up to a problem independent factor defined in the algorithm. The total work for the algorithm with decreasing tolerance turns out to be bounded by $\mathcal{O}(N)$, including all refinement levels; an alternative version with constant tolerance stops with $\mathcal{O}(N \log N)$ total steps. Furthermore, the global error of the algorithm is proven to be asymptotically bounded by the specified error tolerance.

Similar results on convergence rates of adaptive algorithms with either stochastic or deterministic time steps for weak approximations of Itô stochastic differential equations are proved for the Monte Carlo Euler method. The main step in this extension is the proof of the almost sure convergence of the error density.

Numerical experiments illustrate the behavior of the adaptive algorithms and show by carefully chosen examples the efficiency to use the adaptive methods compared to a uniform step size or a method based only on the local error (without the weight).

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Key words and phrases. adaptive mesh refinement algorithm, computational complexity, a posteriori error estimate, ordinary differential equations, stochastic differential equations, Monte Carlo method, almost sure convergence.

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Preface

This thesis presents convergence rates of adaptive algorithms for deterministic and Itô stochastic differential equations, derived in

Paper 1 K.-S. Moon, A. Szepessy, R. Tempone and G.E. Zouraris, *Convergence rates for adaptive approximation of ordinary differential equations based on global and local errors*, preprint, 2001, and

Paper 2 K.-S. Moon, A. Szepessy, R. Tempone and G.E. Zouraris, *Convergence rates for adaptive weak approximation of stochastic differential equations*, preprint, 2001.

The work included in this thesis has been carried out from April 1999 to October 2001 at the department of Numerical Science and Computer Science (NADA), Royal Institute of Technology (KTH), in Stockholm, Sweden.

First of all, I would like to express my deepest gratitude to my advisor, prof. Anders Szepessy, for continuous encouragement and all the support throughout this work. I would also like to thank Raúl Tempone and Dr. Georgios E. Zouraris for many valuable discussions about mathematics as well as life.

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Stockholm, 2001*

Chapter 1

Introduction to Adaptive Algorithms

Several mathematical models are formulated by deterministic or stochastic differential equations, ranging from physics to finance. Quantitative study of solutions of differential equations typically requires numerical approximations. As for all reliable measurements, a useful computation requires an estimate of its accuracy. Higher accuracy can generally only be achieved by increasing the computational work. Thus adaptive methods combine the two goals, to control the approximation error and to minimize the computational work, in some sense. There are numerous adaptive algorithms for differential equations, however the theoretical understanding of optimal convergence rates of adaptive algorithms is not as well developed. This work presents convergence rates of adaptive algorithms for *ordinary differential equations* (ODEs) and for weak approximation of *Itô stochastic differential equations* (SDEs), based on global and local errors.

A differential equation is an equation in which the derivatives of functions of independent variables appear. The general solution of an p -th order ODE contains p free parameters, which can be uniquely determined by prescribed initial conditions. Since the time of Newton (1642-1727), differential equations have become an essential mathematical tool for the solution of many physical problems, see [9], [16], [17]. A stochastic differential equation allows some randomness in the input functions defining the equation. The solution of a SDE is a stochastic process, which is a mathematical model for the occurrence of a random phenomenon. The applications of stochastic processes are constantly growing including finance, signal and image processing and environmental science, see [23], [24], [31], [35]. The quantities that we calculate in stochastic models are mathematical expectations. The most efficient and widely applicable approach to compute expectations is the simulation of sample paths of time discrete approximations using an imitation of

randomness, called Monte Carlo method. If we suppose the imitation of randomness is perfectly realized, the Monte Carlo method is absolutely rigorous, which allows us to establish results by probabilistic reasoning, see [12], [25], [29], [31].

The goal of adaptive algorithms for differential equations is to solve problems with an adjusted mesh using as little computational work as possible, for a given level of accuracy. The main ingredient of adaptive algorithms is an error estimate to measure its accuracy. For a deterministic case, error estimates based on local error can be derived by the classical error equation and linearization, see [9], [18], by Galerkin orthogonality using either local problems or the residual, see [1], [13] and by a variational principle, following Alekseev [2] and Gröbner [15]. The error estimate in this work, derived by a variational principle, has the form

$$\text{Global error} = \sum_{\text{time steps}} \text{local error} \cdot \text{weight}, \quad (1.1)$$

which is applicable to general discretization of differential equations. For a stochastic Itô differential equation, the classical linearization technique, based on the direct comparison between two different realizations, is not relevant for weak convergence since the realizations might not be close. Talay and Tubaro [33] developed an error expansion in a priori form for the weak approximation using the Monte Carlo Euler method. Inspired by this results, the work [32] derived new expansions of the computational error with computable leading order term in a posteriori form, based on stochastic flows and discrete dual backward problems.

Based on the error estimates derived in [27] and [32] for ODEs and SDEs respectively, the global error has an error representation by

$$\text{Global error} = \sum_{\text{time steps}} \text{error indicator}. \quad (1.2)$$

Then a typical adaptive algorithm does two things iteratively :

- (1) if the error indicators satisfy an accuracy condition, then it stops; otherwise
- (2) the algorithm chooses where to refine the mesh and then makes an iterative step to (1).

Therefore the indicators not only estimate the global error, but also give information to refine for an optimal mesh.

An approximation of the weight function in (1.1), which solves a certain linear backward dual problem obtained by linearizing a given forward problem around the approximate solution, plays an important role for step (2). Therefore, the algorithm use additional computational work to obtain the error estimate, which also inform where to refine the mesh to achieve efficiency. The use of dual functions is standard in optimal control theory and in particular for adaptive mesh control for ordinary and partial differential equations, see [3], [4], [5], [6], [7], [14], [21], [22], [26], [34].

Despite the wide use of adaptive algorithms and well developed theory of a posteriori error estimates, little is known theoretically on the behavior of adaptive mesh refinement algorithms; there are however recent important contributions. Devore studies in [11] the efficiency of nonlinear approximation of functions, including wavelet expansions, based on smoothness conditions in Besov space. Inspired by this approximation results, Cohen, Dahmen and Devore prove in [8] that a wavelet-based adaptive N -term approximation produces an approximation with optimal error $\mathcal{O}(N^{-s})$ in the energy norm for linear coercive elliptic problems, see also [10]. For strong approximation of stochastic differential equations, Hofmann, Müller-Gronbach and Ritter prove in [19], [20] and [30] optimal convergence rates of adaptive and uniform step size control.

The main result of this work is construction and analysis of adaptive algorithms including dividing and merging of time steps for ODEs and weak approximation of SDEs. In particular, we prove that

- The global error is asymptotically bounded by the specified error tolerance.
- The adaptive algorithms stop with the optimal number of steps up to a problem independent factor.

These results on efficiency and accuracy of the adaptive algorithms are based on the convergence of error density, which measures the approximation error for each interval. In particular, *almost sure* (*a.s.*) convergence of error density for weak approximation is a consequence of the *a.s.* convergence of the approximate solution as maximal step size tends to zero.

In this thesis, Chapter 2 describes a deterministic problem and summarizes the results including the error estimate and the adaptive algorithm in [27]. Chapter 3 introduces a stochastic problem and analyzes the adaptive algorithms with stochastic and deterministic time steps based on *a.s.* convergence of the error density in [28].

Chapter 2

Convergence rates for ODEs

Consider a solution $X : [0, T] \rightarrow \mathbb{R}^d$ of a differential equation, with a given flux $a : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$\begin{aligned}\frac{dX(t)}{dt} &= a(t, X(t)), \quad 0 < t \leq T, \\ X(0) &= X_0,\end{aligned}\tag{2.1}$$

and an approximation \bar{X} of (2.1) by any p -th order accurate numerical method, satisfying the same initial condition $\bar{X}(0) = X(0) = X_0$ with time steps $0 = t_0 < \dots < t_N = T$.

The goal is to derive error estimates of the form (1.1) where

$$\text{global error} \equiv g(X(T)) - g(\bar{X}(T)),$$

and $g : \mathbb{R}^d \rightarrow \mathbb{R}$ is a given general function. The estimates will use the local error e defined by $e(t_n) \equiv \tilde{X}(t_n) - \bar{X}(t_n)$, where the local exact solution \tilde{X} satisfies for each time step $(t_{n-1}, t_n]$

$$\begin{aligned}\frac{d\tilde{X}(t)}{dt} &= a(t, \tilde{X}(t)), \\ \tilde{X}(t_{n-1}) &= \bar{X}(t_{n-1})\end{aligned}\tag{2.2}$$

and the weight function Ψ satisfies for $t < s < T$ the dual equation

$$\begin{aligned}-\frac{d\Psi(s)}{ds} &= (a')^*(s, X(s))\Psi(s), \\ \Psi(T) &= g'(X(T)),\end{aligned}\tag{2.3}$$

where $(a')^*(s, x)$ is the transpose of the Jacobian matrix. Based on the definitions (2.2) and (2.3), we derive in Theorem 2.1 in [27] an error representation,

$$g(X(T)) - g(\bar{X}(T)) = \sum_{n=1}^N \left(e(t_n), \int_0^1 \Psi(t_n, \bar{X}(t_n) + se(t_n)) ds \right)\tag{2.4}$$

where (\cdot, \cdot) is the standard scalar product on \mathbb{R}^d . This error indicator is unique in some sense, see Theorem 3.1.

However in (2.4) both the true local errors and the weights are non computable, therefore the idea in the next step is to transfer the representation (2.4) to the following expansion

$$g(X(T)) - g(\bar{X}(T)) = \sum_{n=1}^N (\bar{e}(t_n), \bar{\Psi}(t_n)) + \int_0^T o(\Delta t^p(t)) dt. \quad (2.5)$$

Here the approximate local error \bar{e} is defined by $\bar{e}(t_n) \equiv \gamma(\bar{\bar{X}}(t_n) - \bar{X}(t_n))$, where γ is a Richardson extrapolation constant and $\bar{\bar{X}}$ is obtained with smaller time steps or a higher order method than \bar{X} and the approximation $\bar{\Psi}$ is a consistent approximation of Ψ .

Based on the error expansion in (2.5) and a given bound on the global error, the number of time steps are minimized by choosing for all time steps

$$\text{local error} \cdot \text{weight} = \text{constant}. \quad (2.6)$$

This motivates that the adaptive algorithm starts with an initial partition $\Delta t[1]$ and then specifies iteratively a new partition $\Delta t[k+1]$, from $\Delta t[k]$, using the following dividing and merging strategy, see Figure 2.1: for $n = 1, 2, \dots, N[k]$ let $\bar{r}_n[k] \equiv |(\bar{e}(t_n), \bar{\Psi}(t_n))|$ and

$$\mathbf{if} \ \bar{r}_n[k] > s_1 \frac{\text{TOL}}{N}, \ \text{then divide } \Delta t_n[k] \text{ into } M \text{ sub steps} \quad (2.7)$$

$$\mathbf{elseif} \ \max(\bar{r}_n[k], \bar{r}_{n+1}[k]) < s_2 \frac{\text{TOL}}{N}, \ \text{then merge } \Delta t_n[k] \text{ and} \quad (2.8)$$

$\Delta t_{n+1}[k]$ into one step, and increase n by 1,

else let the new step be the same as the old.

endif

until it satisfies the following stopping criteria:

$$\mathbf{if} \ \left(\bar{r}_n[k] \leq S_1 \frac{\text{TOL}}{N}, \ \forall n = 1, \dots, N \right) \text{ and} \quad (2.9)$$

$$\left(\max(\bar{r}_n[k], \bar{r}_{n+1}[k]) \geq S_2 \frac{\text{TOL}}{N}, \ \forall n = 1, \dots, N-1 \right) \quad (2.10)$$

then we stop.

Here M is a given integer greater than 1, which bounds the increment of the number of time steps from one iteration to the next.

The choice of the parameters $S_2 < s_2 < s_1 < S_1$ is based on the understanding of the variation of the error density, $\bar{\rho}(t) = (\bar{e}(t_n), \bar{\Psi}(t_n)) / \Delta t_n^{p+1}$, $t \in (t_{n-1}, t_n)$, at

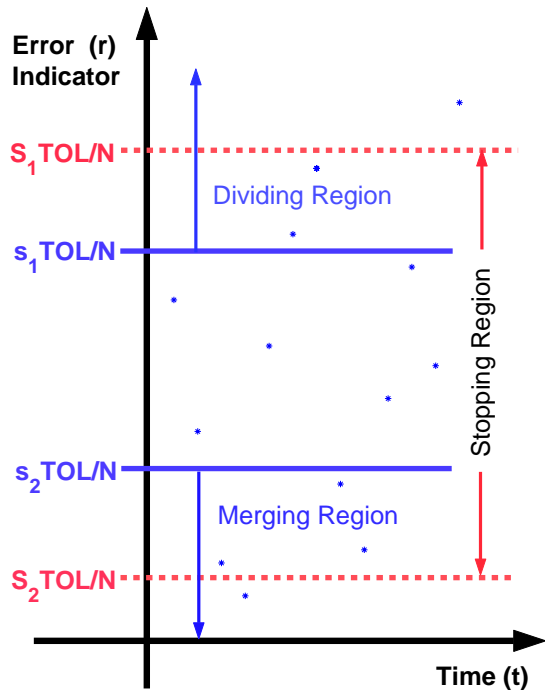


Figure 2.1. Dividing and merging time steps of the adaptive algorithm.

two consecutive iterations using any p -th order accurate method. Let us assume that for sufficiently small TOL there exists a constant $c > 0$ such that for all $t \in [0, T]$

$$c \leq \left| \frac{\bar{\rho}(t)[k+1]}{\bar{\rho}(t)[k]} \right| \leq c^{-1}, \quad (2.11)$$

based on the fact that the error density converges as the tolerance parameter tends to zero. Then the adaptive algorithm is proven to stop, see Theorem 3.2, by the stopping criteria (2.9, 2.10) after a finite number of operations, provide the constants satisfy for $c \geq 2^{-p}$,

$$s_2 \leq \frac{1}{2} c M^{-(p+1)} s_1, \quad S_2 < \frac{c}{2} s_2, \quad S_1 > \frac{M}{c} s_1. \quad (2.12)$$

Once the algorithm is proved to stop, we can analyze the quality of the approximate solution. In view of accuracy, Theorem 3.3 shows that the global error is bounded by TOL asymptotically, i.e.,

$$\limsup_{\text{TOL} \rightarrow 0^+} \text{TOL}^{-1} |g(X(T)) - g(\bar{X}(T))| \leq S_1, \quad (2.13)$$

using the upper bound (2.9) of the error indicators and the convergence of $\bar{\rho}$. For a p -th order accurate method, the number of uniform steps to approximate with a given error turns out to be

$$N^{\text{uni}} = \frac{T}{\text{TOL}^{\frac{1}{p}}} \|\bar{\rho}\|_{L^1}^{\frac{1}{p}}, \quad (2.14)$$

while the optimal number of adaptive steps is

$$N^{\text{opt}} = \frac{1}{\text{TOL}^{\frac{1}{p}}} \|\bar{\rho}\|_{L^{\frac{1}{p+1}}}^{\frac{1}{p}}. \quad (2.15)$$

Therefore an adaptive method uses less time steps than a uniform method, due to Jensen's inequality $\|f\|_{L^{\frac{1}{p+1}}} \leq T^p \|f\|_{L^1}$. In view of efficiency, Theorem 3.4 shows that the algorithm stops with the optimal number, N , of steps, i.e.,

$$\text{TOL}^{\frac{1}{p}} N \leq C \|\bar{\rho}\|_{L^{\frac{1}{p+1}}}^{\frac{1}{p}}, \quad (2.16)$$

where C is bounded by $(2^{p+1}/s_2)^{\frac{1}{p}}$ asymptotically. Note that the number of final time steps is related to the numerical effort needed to compute the approximation. To be more precise, the total number of time steps, including all refinement levels, can be bounded by $\mathcal{O}(N \log N)$ for a constant tolerance; an alternative version with decreasing tolerance stops with $\mathcal{O}(N)$ total steps, see Theorem 3.6.

Chapter 3

Convergence rates for SDEs

Consider a stochastic process $(X(t; \omega))$ satisfying the following Itô stochastic differential equation

$$dX_k(t) = a_k(t, X(t))dt + \sum_{\ell=1}^{\ell_0} b_k^\ell(t, X(t))dW^\ell(t), \quad t > 0, \quad k = 1, \dots, d \quad (3.1)$$

and an approximation $\bar{X}(t_n)$ by Euler Monte Carlo method, which is a time discretization based on the nodes $0 = t_1 < t_2 < \dots < t_{N+1} = T$, satisfying

$$\bar{X}(t_{n+1}) - \bar{X}(t_n) = \Delta t_n a(t_n, \bar{X}(t_n)) + \sum_{\ell=1}^{\ell_0} \Delta W_n^\ell b^\ell(t_n, \bar{X}(t_n)) \quad (3.2)$$

with the time increments $\Delta t_n \equiv t_{n+1} - t_n$ and the Wiener increments $\Delta W_n^\ell \equiv W^\ell(t_{n+1}) - W^\ell(t_n)$. Here the randomness is generated by the independent one dimensional Wiener processes $W^\ell(t; \omega)$, $\ell = 1, \dots, \ell_0$, on the probability space (Ω, \mathcal{F}, P) , and the functions $a(t, x) \in \mathbb{R}^d$ and $b^\ell(t, x) \in \mathbb{R}^d$, $\ell = 1, \dots, \ell_0$, are given drift and diffusion fluxes.

For a given general function $g : \mathbb{R}^d \rightarrow \mathbb{R}$, the goal of the adaptive algorithms is to construct the numerical approximation using as few time steps N and realizations M as possible, with the approximation error bounded by a given error tolerance TOL, i.e. the event

$$\left| E[g(X(T))] - \frac{1}{M} \sum_{j=1}^M g(\bar{X}(T; \omega_j)) \right| \leq \text{TOL} \quad (3.3)$$

has a probability close to one. The computational error in (3.3) naturally separates into the time discretization error and the statistical error

$$\begin{aligned} & E[g(X(T))] - \frac{1}{M} \sum_{j=1}^M g(\bar{X}(T; \omega_j)) \\ &= (E[g(X(T)) - g(\bar{X}(T))]) + \left(E[g(\bar{X}(T))] - \frac{1}{M} \sum_{j=1}^M g(\bar{X}(T; \omega_j)) \right), \end{aligned} \quad (3.4)$$

which determines the time steps and the number, M , of realizations respectively. The statistical error is approximated by $c_0 \mathcal{S}(g(\bar{X}(T))) / \sqrt{M}$ from the central limit theorem, where $\mathcal{S}(g(\bar{X}(T)))$ is the sample standard deviation of $g(\bar{X}(T))$ and c_0 is a constant defined in the choice of $M \simeq (c_0 \mathcal{S}(g(\bar{X}(T))) / \text{TOL})^2$. The time discretization error is estimated by the a posteriori error expansions derived in [32].

Based on this error approximation, we develop two adaptive algorithms including dividing and merging of time steps, either with stochastic or with deterministic time steps. The difference between the two algorithms is that the stochastic time steps may use different meshes for all realizations, see Figure 3.1 and the deterministic time steps uses the same mesh for all realizations. The deterministic time steps are useful for solutions with singularities at deterministic times or for problems with small noise, on the other hand, the stochastic time steps are advantageous for problems with singularities at random times, see Section 5 in [28].

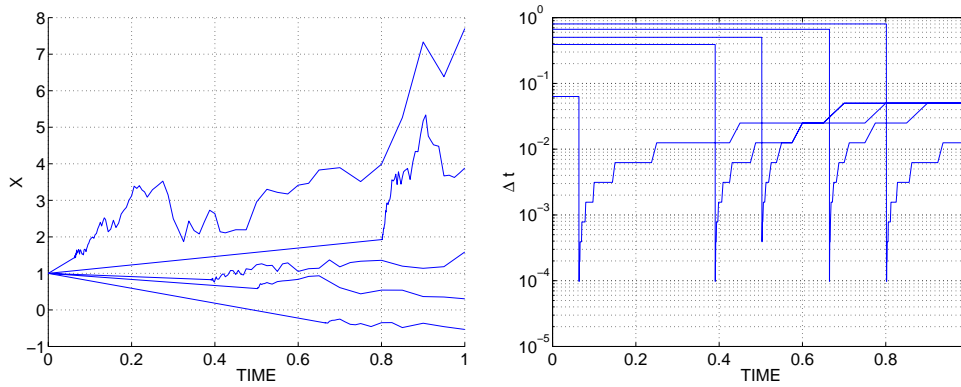


Figure 3.1. Trajectories and the corresponding mesh size for five realizations of solutions by adaptive algorithm with stochastic time steps for singularities at random times.

For the stochastic time steps, the refinement strategy and the stopping criteria are similar to (2.7 - 2.10) except that the algorithm needs to iterate the procedures for each realization until the statistical error is bounded by a given tolerance, since

the error indicators now depend also on the randomly generated realizations. For each realization, Theorem 3.2 shows that the adaptive dividing-merging process stops after a finite number of operations, provided the error density, ρ , satisfies (2.11) for all realizations and $t \in [0, T]$, the increment number, $M = 2$ in (2.12) and for $c \geq 4^{-1}$ the constants $S_2 < s_2 < s_1 < S_1$ satisfy

$$s_2 < \frac{c}{4}s_1, \quad S_2 < cs_2, \quad S_1 > \frac{1}{c}s_1. \quad (3.5)$$

For the deterministic time steps, first we determine the mesh using criteria similar to (2.7 - 2.10) with sample average of error density, then we determine the number, M , of realizations until the statistical error is bounded by a given tolerance, since the mesh is fixed for all realizations. Similar result as (3.5) is proved in Theorem 3.8 for the deterministic case.

There are two main results on efficiency and accuracy of the adaptive algorithms described in [28]. First in view of accuracy, Theorem 3.3 and 3.9 show that, with probability close to one, the approximation errors in (3.3) are asymptotically bounded by the specified error tolerance times a problem independent factor as the tolerance parameter tends to zero, i.e., the approximate error has the estimate

$$\liminf_{\text{TOL} \rightarrow 0} P \left(\frac{|E[g(X(T))] - \frac{1}{M} \sum_{j=1}^M g(\bar{X}(T; \omega_j))|}{\text{TOL}} \leq \frac{S_1}{3} + \frac{2}{3} \right) \geq \int_{-c_0}^{c_0} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

Second in view of efficiency, Theorem 3.4 and 3.10 show that both algorithms with stochastic steps and deterministic steps stop with the optimal expected number of steps and optimal number of steps respectively up to a problem independent factor. For the Monte Carlo Euler method, the relation between the optimal expected number of stochastic steps, $E[N_S]$, optimal number of deterministic steps, N_D , and the constant steps, N_C , is

$$E[N_S] \leq N_D \leq N_C, \quad (3.6)$$

by Jensen's inequality. The number of final time steps is related to the numerical effort to compute the approximation. To be more precise, the total work for deterministic steps is roughly $M \cdot N$, where M is the final number of realizations and N is the final number of time steps, since the work to determine the mesh turns out to be negligible. On the other hand, the total work with stochastic steps is on average bounded by $M \cdot E[N_{\text{tot}}]$, where the total number, N_{tot} , of steps including all levels of refinement is bounded by $\mathcal{O}(N \log N)$ with N steps in the final refinement; for each realization it is necessary to determine a mesh, which may vary for each realization.

The properties of accuracy and efficiency are based on the fact that the error density converges *a.s.* as the error tolerance tends to zero. This convergence is a consequence of the fact that for any $\alpha < 1/2$

$$\lim_{\eta \rightarrow \infty} (\Delta t_{sup}(\eta))^{-\alpha} \sup_{t \in [0, T]} |X(t) - \bar{X}(t)| = 0 \quad a.s., \quad (3.7)$$

with the maximum time step sizes, $\Delta t_{sup}(\eta) \equiv \sup_{1 \leq n \leq N} \Delta t_n(\eta) = T2^{-\eta}$, see Theorem 4.1. In stochastic analysis the concept adapted to W means that a process at time t only depends on the events generated by $\{W(s) : s < t\}$. Although the time steps are not adapted to the standard filtration generated by W for the stochastic time stepping algorithm, the work [32] proved that the corresponding approximate solution converges to the correct adapted solution X , which makes possible to prove (3.7).

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