Quantum Computation - Lecture 11 - The Local Hamiltonian Problem

Mateus de Oliveira Oliveira

TCS-KTH

February 20, 2013
Nondeterministic Polynomial Time (NP)

A language $L \subseteq \{0,1\}^*$ is in $NP$:

- There exists a uniform sequence $C_1, C_2, ...$ of circuits such that
  - If $x \in L$, then there exists $y$ such that $C_1(x, y) = 1$.
  - If $x \not\in L$, then for every $y$, $C_1(x, y) = 0$. 
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$k$-Local Hamiltonian

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**k-Local Hamiltonian**

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  - Each $H_j[S_j]$ is an Hermitian operator acting on a set $S_j$ of qubits.
  - $|S_j| \leq k$.
  - Both $H_j$ and $I - H_j$ are positive semidefinite.
The \( k \)-Local Hamiltonian Problem:

- Given a \( k \)-local Hamiltonian \( H \).
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- and constants \( a, b \) with \( 0 \leq a < b, b - a \geq \frac{1}{p(n)} \) for some polyn. \( p \)
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\( H \) has an eigenvalue not exceeding \( a \).
All eigenvalues of \( H \) are greater than \( b \).
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$k$-local Hamiltonian is in QMA:

- Let $H = \sum_{i=1}^{r} H_{i}[S_{i}]$

\[\text{Construct a circuit } C \text{ such that when applied to state } |\eta\rangle \in B^{n} \text{ the probability of returning } 1 \text{ is } p = 1 - r - 1 \langle \eta | H | \eta \rangle.\]

- If there exists $|\eta\rangle$ with eigenvalue $\leq a$, \[p = 1 - r - 1 \langle \eta | H | \eta \rangle = 1 - r - 1 \lambda \geq 1 - r - 1 a.\]

- If every eigenvalue of $H$ exceeds $b$ then \[p = 1 - r - 1 \langle \eta | H | \eta \rangle \leq 1 - r - 1 b.\]
$k$-local Hamiltonian is in QMA:

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\[ H_j = \sum_s \lambda_s |\psi_s\rangle \langle \psi_s| \text{ since } H_j \text{ is positive semidefinite.} \]
• $H_j = \sum_s \lambda_s |\psi_s\rangle \langle \psi_s|$ since $H_j$ is positive semidefinite.

• Since $H_j$ acts on a bounded number of qubits, it can be realized by the unitary:

$$W_j: |\psi_s, 0\rangle \rightarrow |\psi_s\rangle \otimes (\sqrt{\lambda_s} |0\rangle + \sqrt{1 - \lambda_s}) |1\rangle$$
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This unitary acts on a constant number of qubits: \( S_j \) plus an "answer qubit".
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Exercise: The outcome of \( W_j \) is 1 with probability 1 \(- \langle \eta | H_j | \eta \rangle\)
The Local Hamiltonian Problem

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- Putting all together: Apply the operator \( \sum_j |j\rangle \langle j| \otimes W_j \) to the state

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\frac{1}{\sqrt{r}} \sum_j |j\rangle \otimes |\eta, 0\rangle
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\[ \frac{1}{\sqrt{r}} \sum_j |j\rangle \otimes |\eta, 0\rangle \]

Then the probability of getting outcome 1 is

\[ \sum_j \frac{1}{r} (1 - \langle \eta | H | \eta \rangle) = 1 - r^{-1} \langle \eta | H | \eta \rangle \]
Local Hamiltonian is QMA-complete:

- Given a circuit $U = U_L ... U_1$
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- Given a circuit $U = U_L \ldots U_1$
- Define a Hamiltonian $H = H_{in} + H_{prop} + H_{out}$ acting on $N$ qubits + a clock register on $L + 1$ qubits.

The vector $|\eta\rangle$ minimizing $\langle \eta | H | \eta \rangle$ will be:

$|\eta\rangle = \frac{1}{\sqrt{L + 1}}$

In other words $|\eta\rangle$ will encode the whole history of execution of the quantum circuit.

$H_{in}$ corresponds to the condition that at step 0, all the qubits, but $m$ are in state $|0\rangle$.

$H_{in} = (N \sum_{s=m+1}^N \Pi_s(1)) \otimes |0\rangle \langle 0|$.

Here $\Pi_s(b)$ is the projection onto the space where the $s$-th qubit is $b$.

The term $H_{in}$ adds a penalization of 1 to the function $\langle \eta | H | \eta \rangle$ whenever the qubit $s$ is in state $|1\rangle$ while the counter is in state $|0\rangle$. 
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- Here $\Pi_s^{(b)}$ is the projection onto the space where the $s$-th qubit is $b$.
- The term $H_{in}$ adds a penalization of 1 to the function $\langle \eta | H | \eta \rangle$ whenever the qubit $s$ is in state $|1\rangle$ while the counter is in state $|0\rangle$. 
$H_{out} = \Pi_{1}^{(0)} \otimes |L\rangle\langle L|$
\[ H_{out} = \Pi^{(0)}_1 \otimes |L\rangle\langle L| \]

Assume that the output qubit is the qubit number 1
$H_{out} = \Pi_1^{(0)} \otimes |L\rangle\langle L|$  

Assume that the output qubit is the qubit number 1  

Add a penalization whenever the qubit 1 is $|0\rangle$ in the end of the computation.
\[ H_{\text{prop}} = \sum_{j=1}^{L} H_j \]
The Local Hamiltonian Problem

\[ H_{\text{prop}} = \sum_{j=1}^{L} H_j \]

\[ H_j = -\frac{1}{2} U_j \otimes |j\rangle\langle j - 1| - \frac{1}{2} U_j^\dagger \otimes |j - 1\rangle\langle j| + \frac{1}{2} I(|j\rangle\langle j| + |j - 1\rangle\langle j - 1|) \]
\( H_{prop} = \sum_{j=1}^{L} H_j \)

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- \( H_{prop} \) penalizes a wrong propagation.
The Local Hamiltonian Problem

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- $H_{prop}$ penalizes a wrong propagation.

- Each term $H_j$ corresponds to a transition from $j - 1$ to $j$. 
Change of Basis:

\[ W = \sum_{j=0}^{L} U_j \ldots U_1 \otimes |j\rangle\langle j| \]

\( W \) is a measurement operator that respects the value of the counter \( |j\rangle \).

The vector \( |\eta\rangle \) corresponding to the propagation will be equal to \( W |\tilde{\eta}\rangle \).

Let's see the action of \( W \) on the Hamiltonian \( H \):

\[ \tilde{H} = W^\dagger HW \]

\[ \tilde{H}^{\text{in}} = W^\dagger H^{\text{in}} W \]

\[ \tilde{H}^{\text{out}} = W^\dagger H^{\text{out}} W = (U_j^\dagger \Pi(0) U_j) \otimes |j\rangle\langle j| \]

\[ \tilde{H}^{\text{prop}} = W^\dagger H^{\text{prop}} W \]

\[ \sum_j W^\dagger H_j W = \sum_j W^\dagger \times W \]

\[ \star \]

\[ W^\dagger H_j W = I \otimes |j\rangle\langle j| - 1 \]

\[ \star \]

Thus \( W^\dagger H_j W = I \otimes E_j \) where

\[ E_j = \frac{1}{2}(|j-1\rangle\langle j-1| - |j-1\rangle\langle j| - |j-1\rangle\langle j| - |j-1\rangle\langle j-1|) \]

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where

\[ E = \sum_j E_j \]
Change of Basis:

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- Let's see the action of \( W \) on the Hamiltonian \( H \):

\[
\begin{align*}
W^\dagger H W &= W^\dagger H_{\text{in}} W = \left(U_1^\dagger \Pi (0) U_1 \right) \otimes |L\rangle\langle L| \\
W^\dagger H_{\text{prop}} W &= \sum_j W^\dagger U_j H_j W = I \otimes |j-1\rangle\langle j-1| \quad \text{(Check!)}
\end{align*}
\]

Thus \( W^\dagger H_j W = I \otimes E_j \) where \( E_j = \frac{1}{2}( |j-1\rangle\langle j-1| - |j-1\rangle\langle j-1| - |j\rangle\langle j-1| + |j\rangle\langle j-1|) \)
Change of Basis:

- $W = \sum_{j=0}^{L} U_j \cdots U_1 \otimes |j\rangle\langle j|$
- $W$ is a measurement operator that respects the value of the counter $|j\rangle$.
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  - $\tilde{H}_{in} = W^\dagger H_{in} W = H_{in}$
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  - \( \tilde{H}_{out} = W^\dagger H_{out} W = (U^\dagger \Pi_1^{(0)} U) \otimes |L\rangle\langle L| \)
Change of Basis:

- \( \mathcal{W} = \sum_{j=0}^{L} U_j \ldots U_1 \otimes |j\rangle\langle j| \)
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- The vector \( |\eta\rangle \) corresponding to the propagation will be equal to \( \mathcal{W}|\tilde{\eta}\rangle \).
- Let's see the action of \( \mathcal{W} \) on the Hamiltonian \( \mathcal{H} \):
  - \( \tilde{\mathcal{H}} = \mathcal{W}^\dagger \mathcal{H} \mathcal{W} \)
  - \( \tilde{\mathcal{H}}_{\text{in}} = \mathcal{W}^\dagger \mathcal{H}_{\text{in}} \mathcal{W} = \mathcal{H}_{\text{in}} \)
  - \( \tilde{\mathcal{H}}_{\text{out}} = \mathcal{W}^\dagger \mathcal{H}_{\text{out}} \mathcal{W} = (U^\dagger \Pi_1^{(0)} U) \otimes |L\rangle\langle L| \)
  - \( \tilde{\mathcal{H}}_{\text{prop}} = \mathcal{W}^\dagger \mathcal{H}_{\text{prop}} \mathcal{W} = \sum_j \mathcal{W}^\dagger H_j \mathcal{W} \)
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  - \( \tilde{H} = W^\dagger HW \)
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  - \( \tilde{H}_{out} = W^\dagger H_{out} W = (U^\dagger U_1^{(i)}) \otimes |L\rangle \langle L| \)
  - \( \tilde{H}_{prop} = W^\dagger H_{prop} W = \sum_j W^\dagger H_j W \)
    - \( W^\dagger (U_j \otimes |j\rangle \langle j - 1|) W = I \otimes |j\rangle \langle j - 1| \) (Check!)
Change of Basis:

- \( W = \sum_{j=0}^{L} U_j \cdots U_1 \otimes |j\rangle \langle j| \)
- \( W \) is a measurement operator that respects the value of the counter \( |j\rangle \).
- The vector \( |\eta\rangle \) corresponding to the propagation will be equal to \( W|\tilde{\eta}\rangle \).

Let's see the action of \( W \) on the Hamiltonian \( H \):

\[
\begin{align*}
\tilde{H} &= W^\dagger H W \\
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&\quad \quad \bullet \quad W^\dagger (U_j \otimes |j\rangle \langle j - 1|) W = I \otimes |j\rangle \langle j - 1| \text{ (Check!)} \\
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    - Thus \( W^\dagger H_j W = I \otimes E_j \) where
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      E_j = \frac{1}{2} (|j - 1\rangle \langle j - 1| - |j - 1\rangle \langle j - 1| - |j\rangle \langle j - 1| + |j\rangle \langle j|) \]
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  - \( W^\dagger H_{prop} W = I \otimes E \) where \( E = \sum_j E_j \)
If the answer is YES there is a small eigenvalue:

- Suppose the original circuit gives the answer YES with probability greater than $1 - \varepsilon$ on some input vector $|\chi\rangle$
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- Finally

$$\langle \tilde{\eta} | \tilde{H}_{out} |\tilde{\eta}\rangle = \langle \tilde{\eta} | (U^\dagger \Pi_1^{(0)} U \otimes |L\rangle\langle L|) |\tilde{\eta}\rangle = P(0) \frac{1}{L+1} \leq \frac{\varepsilon}{L+1}$$