Quantum Computation - Lecture 10 - Approximation of the Jones Polynomial

Mateus de Oliveira Oliveira

TCS-KTH

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Knots
• Knots
• Invariants for knots
Knots

Invariants for knots

The Jones Polynomial
Overview

Knots

Invariants for knots

The Jones Polynomial

Approximating the JP
To each PLUS-ORIENTED-CROSSING assign $+1$. 
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To each MINUS-ORIENTED-CROSSING assign −1.
To each PLUS-ORIENTED-CROSSING assign $+1$.

To each MINUS-ORIENTED-CROSSING assign $-1$.

How to easily see?

- Rotate the head of the line on the top towards the head of the line on the bottom.
- If the rotation is counterclockwise, assign $+1$.
- If the rotation is clockwise, assign $-1$. 

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State \( \sigma \): Replace each CROSSING-MINUS for \( \{ \bigcup, || \} \).
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  \item \( \sigma(L) \):
\end{itemize}
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  - Kauffman bracket polynomial:

$$\langle L \rangle = \sum_{all\ states \ \sigma} \sigma(L)$$
Jones Polynomial:

\[ V_L(t) = V_L(A^{-4}) = (-A)^{3w(L)} \langle L \rangle \]  \hspace{1cm} (1)

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Jones Polynomial:

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- \( w(L) \) is the writhe
- \( \langle L \rangle \) is the Kauffman Bracket ignoring the orientation.
$r$ dimensional representation $\Phi$ of an algebra: Linear mapping from the algebra into the set of $r \times r$ complex matrices $M_r$, such that $\Phi(XY) = \Phi(X)\Phi(Y)$. 

Obs: If an algebra is specified by a set of generators, then the representation may be specified by the images of the generators. In that case these images should satisfy the same relations.
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Let \( n \in \mathbb{N} \) and \( d \in \mathbb{C} \) then the Temperley-Lieb algebra \( TL_n(d) \) is the algebra generated by \( \{1, E_1, \ldots, E_{n-1}\} \) with relations

- \( E_i E_j = E_j E_i \), \( |i - j| \geq 2 \)
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\[ \psi(E_i) = \]

\[
\begin{array}{c}
\hline
& & & \\
1 & i & i+1 & n \\
\hline
& & & \\
\end{array}
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Then $\rho_A : B_n \to TL_n(d)$ where $\rho_A(\sigma_i) = AE_i + A^{-1}I$ is a representation of $B_n$ into $TL_n(d)$. 

Exercise:

$\rho_A(\sigma_j) \rho_A(\sigma_j) = I$ for $|i - j| > 1$

$\rho_A(\sigma_i) \rho_A(\sigma_i+1) \rho_A(\sigma_i) = \rho_A(\sigma_i+1) \rho_A(\sigma_i) \rho_A(\sigma_i+1)$

Continuing, it is possible to represent $B_n$ via unitary matrices. Given a representation $\tau$ of $TL_n(d)$ such that $\tau(E_i) = \tau(E_i)^\dagger$ for each of the generators $E_i$.

Exercise: Show that $\tau(\rho_A(\sigma_i)) \tau(\rho_A(\sigma_i))^\dagger = I$.
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A tangle is a braid in which some of the crossings have been replaced by a picture with the form $\bigcup \bigcap$. 
From braids to Knots.
A linear function from an algebra to the complex numbers is called a trace if it satisfies the equation \( tr(XY) = tr(YX) \) for every elements \( X, Y \) in the algebra.

\[
tr( ) = d^{-4} \quad = d^{-2}
\]
A linear function from an algebra to the complex numbers is called a trace if it satisfies the equation $tr(XY) = tr(YX)$ for every elements $X, Y$ in the algebra.

Markov trace $tr : gTL_n(d) \to \mathbb{C}$ is defined on a Kauffman $n$-diagram $K$ as follows.

- Connect the $n$ top frontier points to the $n$ bottom frontier points with non-intersecting curves.
- If $a$ is the number of circles resulting from this operation, then $tr(K) = d^{-a} - n$.

By the isomorphism between $TL_n(d)$ and $gTL_n(d)$ this operation also induces a trace on $TL_n(d)$. 

\[
tr(\text{\includegraphics[width=0.3\textwidth]{diagram1.png}}) = d^{-4} \\
\text{\includegraphics[width=0.3\textwidth]{diagram2.png}} = d^{-2}
\]
• A linear function from an algebra to the complex numbers is called a trace if it satisfies the equation $tr(XY) = tr(YX)$ for every elements $X, Y$ in the algebra.

• Markov trace $tr : gTL_n(d) \rightarrow \mathbb{C}$ is defined on a Kauffman $n$-diagram $K$ as follows.

$tr(\text{Diagram 1}) = d^{-4}$

$= d^{-2}$
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By the isomorphism between \( TL_n(d) \) and \( gTL_n(d) \) this operation also induces a trace on \( TL_n(d) \).
\( tr(1) = 1 \)

Lemma: There is a unique linear map on \( TL_n(d) \) that satisfies the three properties listed above.
\begin{itemize}
  \item $tr(1) = 1$
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Lemma: There is a unique linear map on $TL_n(d)$ that satisfies the three properties listed above.
\begin{itemize}
  \item $tr(1) = 1$
  \item $tr(XY) = tr(YX)$ for any $X, Y \in TL_n(d)$
  \item If $X \in TL_{n-1}(d)$ then $tr(XE_{n-1}) = \frac{1}{d} tr(X)$
\end{itemize}

Lemma: There is a unique linear map on $TL_n(d)$ that satisfies the three properties listed above.
Given a braid $B$, let $B^{tr}$ denote its trace closure. Then

$$V_{B^{tr}}(A^{-4}) = (-A)^{3w(B^{tr})} d^{n-1} \text{tr}(\rho_A(B))$$
Let $|\alpha\rangle$ be a quantum state that can be efficiently prepared.
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• Let $Q$ be a unitary gate that can be applied efficiently.
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Quantum circuit whose output is a random variable $\{-1, 1\}$ and whose expectation is $Re\langle\alpha|Q|\alpha\rangle$. 

1. Start with the state $\sqrt{2}\left(|0\rangle + |1\rangle\right)\otimes|\alpha\rangle$.
2. Apply $Q$ conditioned on the first qubit: $\sqrt{2}\left(|0\rangle\otimes|\alpha\rangle + |1\rangle\otimes Q|\alpha\rangle\right)$.
3. Apply a Hadamard gate on the first qubit: $\sqrt{2}|0\rangle\otimes(|\alpha\rangle + Q|\alpha\rangle) + \sqrt{2}|1\rangle\otimes(|\alpha\rangle - Q|\alpha\rangle)$.
4. Output 1 if the result is $|0\rangle$ and Output $-1$ if the result is $-1$.

Exercise: The expectation of the output is $Re\langle\alpha|Q|\alpha\rangle$.

Exercise: What is the expectation of the output if we start with the state $\sqrt{2}\left(|0\rangle - i|1\rangle\right)\otimes|\alpha\rangle$?
Let $|\alpha\rangle$ be a quantum state that can be efficiently prepared.

Let $Q$ be a unitary gate that can be applied efficiently.

Quantum circuit whose output is a random variable $\{-1, 1\}$ and whose expectation is $\text{Re}\langle\alpha|Q|\alpha\rangle$.

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Do the same for the random variables $y_j$ whose expectation value is $Im\langle \alpha | Q(B) | \alpha \rangle$ using an appropriated version of the Hadamard test.