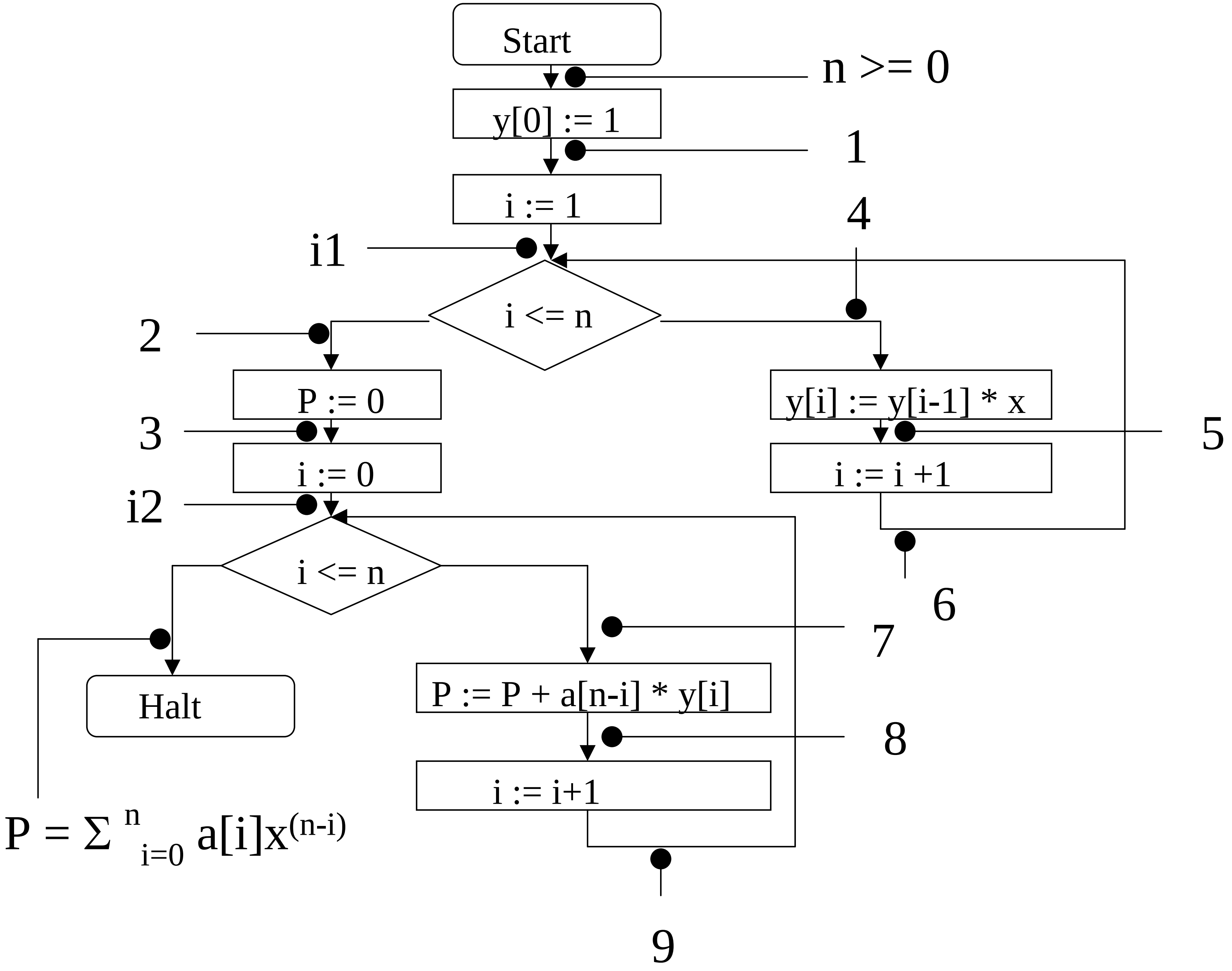
OOTeori 2006
Hemtenta 06-05-17

- Question 1 Model answer

First we need to draw a flowchart, add the pre and postconditions and label the assertion points. Then we need to derived invariant formulas at all label points.



$$P = \Sigma^{n}_{i=0}\ a[i]x^{(n-i)}$$

First we analyse the <u>main flow of control</u> through the flowchart.
The precondition is

$$n \geq 0 \quad \textbf{(precondition)}$$

Adding a tautology gives

$$n \geq 0 \;\&\; 1 = 1$$

By the forwards assignment rule (since the lhs does not occur in the rhs) we get formula 1

$$n \geq 0 \;\&\; y[0] = 1 \; \textbf{(formula 1)}$$

Again adding a tautology,

$$n \geq 0 \;\&\; y[0] = 1 \;\&\; 1=1$$

And (since the lhs does not occur in the rhs) by the forwards assignment rule we get the following formula

$$n \geq 0 \;\&\; y[0] = 1 \;\&\; i=1 \; \textbf{(formula *)}$$

We <u>guess</u> the loop invariant i1 to be

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; n \geq 0 \;\&\; n+1 \geq i \; \textbf{(formula i1)}$$

which obviously follows from formula (*) the first time we encounter the loop.

From i1 and the forwards branching rule and a little arithmetic we get

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; n \geq 0 \;\&\; n+1 \geq i \;\&\; i > n$$

Applying some laws of integer arithmetic gives

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; n \geq 0 \;\&\; n+1 = i$$

Hence we get formula 2

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \;\&\; n \geq 0 \quad \textbf{(formula 2)}$$

Adding a tautology to formula 2 gives

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \;\&\; n \geq 0 \;\&\; 0 = 0$$

Then applying the forwards assignment rule (since the lhs doesn’t occur on the rhs) gives

formula 3

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \;\&\; n \geq 0 \;\&\; P = 0 \quad \textbf{(formula 3)}$$

Adding a tautology to formula 3 gives

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \;\&\; n \geq 0 \;\&\; P = 0 \;\&\; 0 = 0$$

Then applying the forwards assignment rule gives

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \;\&\; n \geq 0 \;\&\; P = 0 \;\&\; i = 0 \quad \textbf{(formula *)}$$

We guess the loop invariant i2 to be

$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \ \& \ P = \sum^{i-1}_{j=0} a[n-j]y[j] + 0 \quad \& \ n+1 \geq i$ **(formula i2)**

But this obviously holds the first time we reach the loop using formula (*)

Applying the forwards branching rule to i2 and using a little arithmetic we get

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \ \& \ P = \sum^{i-1}_{j=0} a[n-j]y[j] + 0 \ \& \ n+1 \geq i \ \& \ i > n$$

which gives

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \ \& \ P = \sum^{i-1}_{j=0} a[n-j]y[j] + 0 \quad \& \ n+1 = i$$

Hence

$$\forall j \quad n \geq j \geq 0 \Rightarrow y[j] = x^{j} \ \& \ P = \sum^{n}_{j=0} a[n-j]y[j]$$

So

$$P = \sum^{n}_{j=0} a[n-j] x^{j}$$

From which we get the postcondition

$$P = \sum^{n}_{j=0} a[j]\, x^{n-j} \quad \textbf{(Postcondition)}$$

It only remains to verify that both loop invariants i1 and i2 are valid invariants.

Let us analyse the first loop body. The loop invariant i1 was guessed to be

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; n+1 \geq i \quad \textbf{(formula i1)}$$

By the applying the forwards branching rule to i1 we get

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; n+1 \geq i \;\&\; n \geq i$$

which gives (replacing with a logically equivalent formula)

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; y[i-1] = x^{i-1} \;\&\; n \geq i$$

Now multiplying by x we get formula 4

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; y[i-1] * x = x^{i} \;\&\; n \geq i \quad \textbf{(formula 4)}$$

Using the forwards assignment rule on formula 4 (noticing that the lhs does not occur in the rhs) we get

$$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; y[i] = x^{i} \;\&\; n \geq i$$

Rewriting this formula we get formula 5

$\forall j \quad (i+1)-1 \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; y[(i+1)-1] = x^{(i+1)-1} \;\&\; n+1 \geq i+1$ **(formula 5)**

Applying the forwards assignment rule to formula 5 (noticing that the lhs does occur in the rhs) we get

$$\forall j \quad i-1 \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; y[i-1] = x^{i-1} \;\&\; n+1 \geq i$$

Rewriting this we get formula 6 which is clearly the same as i2.

$\forall j \quad i \geq j \geq 1 \Rightarrow y[j-1] = x^{j-1} \;\&\; n+1 \geq i$ **(formula 6)**

Next we analyse the second loop body. The loop invariant i2 is guessed to be

$$P = \sum^{i-1}_{j=0} a[n-j]y[j] \ \& \ n+1 \geq i \quad \textbf{(formula i2)}$$

By the applying the forwards branching rule to i2 we get

$$P = \sum^{i-1}_{j=0} a[n-j]y[j] \ \& \ n+1 \geq i \ \& \ n \geq i$$

which simplifies to

$$P = \sum^{i-1}_{j=0} a[n-j]y[j] \ \& \ n \geq i$$

Adding a[n-i] * y[i] to both sides and 1 to both sides we get

$$P + a[n-i] * y[i] = \sum^{i-1}_{j=0} a[n-j]y[j] + a[n-i] * y[i] \ \& \ n+1 \geq i+1$$

And simplifying gives formula 7

$$P + a[n-i] * y[i] = \sum^{i}_{j=0} a[n-j]y[j] \ \& \ n+1 \geq i+1 \quad \textbf{(formula 7)}$$

Applying the forward assignment rule to formula 7 gives

$$P = \sum^{i}_{j=0} a[n-j]y[j] \ \& \ n+1 \geq i+1$$

Rewriting this gives formula 8

$$P = \sum_{j=0}^{(i+1)-1} a[n-j]y[j] \ \& \ n+1 \geq i+1$$ **(formula 8)**

Applying the forwards assignment rule to formula 8 gives formula 9

$$P = \sum_{j=0}^{i-1} a[n-j]y[j] \ \& \ n+1 \geq i$$ **(formula 9)**

But this formula is the same as invariant i2. This completes the analysis of both loop bodies, and our proof is now complete.