KTH Computer Science
and Communication

Homework VI, Theoreticians Toolkit 2009/2010

Due on Tuesday March 2 at 15.15. Solutions to many homework problems, including this one, will be available on the internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the internet you may use and hence whenever in doubt contact Johan Håstad. You are, however, allowed to discuss problems in groups with up to three students, but solutions should be handed in individually.

1 (20 p) Consider a random undirected degree-three graph with $2n$ nodes constructed by the following procedure. For each node construct three "placeholders". Pick a uniformly random matching on the resulting $6n$ placeholders and draw an edge between two nodes v and w if one placeholder of v is matched to one placeholder of w. Note that this definition implies that we allow self-loops. Show that, with high probability, such a graph is an edge-expander for small sets. It is enough to prove that there are positive constants a and α such that for any set, S, of size $s < n^a$ the number of edges between S and its complement is at least αs. It is not important what value you get for a but try to get as good a value for α as possible.

2 (30 p) Let us look at the graph given by the hypercube. It has $n = 2^m$ vertices with labels given by all strings in $\{0, 1\}^m$ and two nodes connected iff their labels differ in exactly one coordinate. Your task is to study this graph from an expander perspective. It is not constant degree, but fairly sparse in that the degree is $m = \log n$. The fact that the degree is large makes edge-expansion and node-expansion slightly different.

2a (10 p) Estimate to what extent the hypergraph is a node-expander. Construct a set S (of size at most $n/2$) such that it has very few neighbors outside S (fewer than $d|S|$ for any constant d).

2b (10 p) Estimate to what extent the hypergraph is an edge-expander. Try to find a set S (of size at most $n/2$) such that relatively few of the edges adjacent to S has its other endpoint in its complement (fewer than $dn|S|$ such edges for any constant d). Can you prove that your set is optimal in some sense?

2c (10 p) Find the eigenvalues and eigenvectors of the adjacency matrix of the graph of the hypergraph. Do this without trying to find the information on the Internet.