KTH Computer Science
and Communication

Homework I, Theoreticians Toolkit 2009

Due on Tuesday Nov 24 at 15.15. Solutions to many homework problems, including this one, will be available on the internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the internet you may use and hence whenever in doubt contact Johan Håstad. You are, however, allowed to discuss problems in groups with up to three students, but solutions should be handed in individually.

1 (20p) Find as large as possible value of the constant c such that a random graph on n nodes contains a clique with[1] $(c - o(1)) \log n$ nodes.

Here we assume that we have a random graph with n nodes such that each edge exists with probability $1/2$. We are only interested in an asymptotic result and hence when making calculations you may drop lower order terms (but of course you need to motivate why indeed something is of lower order). Fully proving the case $c = 1$ gives a score of 5 points (as it was started in class) and getting the optimal value of c is needed for a full score.

2 Constructing a random 3-Sat formula with n variables and $m = \lceil dn \rceil$ (remember that $\lceil x \rceil$ is the smallest integer larger than x) clauses is done in the following way. Randomly take three different variables (all triples being equally likely). With uniform probability choose one of the eight ways to negate these variables and make them into a clause. Repeat with independent randomness until you have m clauses.

2a (5p) For what value of d is the expected number of satisfying assignment roughly[2] 1? Call this value value d_0.

2b (2p) Prove that the formula is likely (probability $1 - o(1)$) to be unsatisfiable for any constant d such that $d > d_0$.

2c (20p) Prove that the formula remains at least somewhat likely to be unsatisfiable also in the case when d is slightly smaller than d_0. The difficulty of this problem is very much dependent on what we mean by "somewhat likely" and "slightly smaller". The exact formulation to prove to get a full score on this problem is that there is some constant $d_1 < d_0$ such that for $d = d_1$ the probability that the corresponding random formula is satisfiable is at most 1/2. The size of $d_0 - d_1$ does not matter for your score on the problem and the main property of a solution to aim for is a mathematically correct argument.

Hint: A satisfiable formula that does not depend on all its variables has many satisfying assignments.

As a curiosity we may note that the value of d such that such a formula has probability around $1/2$ of being satisfiable is not known but conjectured to be around 4.2.

[1] Please remember that all logarithms in this course are with base 2.
[2] To be more precise $\Theta(1)$.