Modeling Turbulent Boundary Layers by Small Friction

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Abstract

We present evidence that a turbulent boundary layer can be modeled by a slip/small friction force boundary condition, which opens the possibility of computational simulation of turbulent flow without resolving thin no-slip boundary layers requiring quadrillions of mesh points. We perform a sensitivity analysis of a turbulent solution by solving an associated dual linearized problem and discover that relevant stability factors are of moderate size, which shows that the effect on the main flow of perturbing a small friction force to zero, is small. The low sensitivity can be seen as an effect of cancellation present in turbulent flow. On the other hand in laminar flow there is no cancellation with the result that slip cannot model a laminar boundary layer.

1 Computational Turbulence

In recent work [14, 15, 12, 16] we have shown that drag and lift in incompressible slightly viscous turbulent bluff-body flow can be accurately predicted by computational solution of the Navier-Stokes equations with slip/small friction force boundary conditions as a model of a turbulent boundary layer. Our results show that the curse of Prandtl requiring resolution of thin no-slip boundary conditions asking for impossible quadrillions of mesh points [19], can be avoided, which opens new possibilities of computational simulation of turbulent flow in a large variety of applications [17].

We compute solutions of the Navier-Stokes equations using an adaptive finite element method with duality based error control described in detail in [14] and referred to as G2 as an acronym of General Galerkin.

2 Turbulent Separation with Slip/Small Friction

We make a distinction between laminar separation from a laminar boundary layer with no-slip velocity boundary condition considered by Prandtl, and turbulent separation from a turbulent boundary layer modeled by a slip/small friction boundary condition [16].
We motivate the use of slip boundary condition by the fact that the skin friction of a turbulent boundary layer (the tangential force from a no-slip boundary condition), tends to zero with the viscosity, which is supported by both experiment and computation, also indicating that boundary layers in general are turbulent. More generally, we use a friction-force boundary condition as a model of the skin friction effect of a turbulent boundary layer, with a (small) friction coefficient determined by the Reynolds number $Re = \frac{UL}{\nu}$, where $U$ is a representative velocity, $L$ a length scale and $\nu$ the viscosity. The limit case of zero friction with slip then corresponds to vanishing viscosity/very large Reynolds number, while large friction models no-slip of relevance for small to moderately large Reynolds numbers. In mathematical terms we combine the Navier-Stokes equations with a natural (Neumann/Robin type) boundary condition for the tangential stress, instead of an essential (Dirichlet type) condition for the tangential velocity as Prandtl did.

We find quantitative evidence in benchmark problems that the effect on mean-value outputs such as lift and drag of modeling a turbulent boundary layer with a slip/small friction boundary condition, is small in the case of small viscosity. We do this by an a posteriori sensitivity analysis by computational solution of a dual problem linearized at a turbulent solution with no-slip and discovering that relevant stability factors are of moderate size, which we understand to be an effect of cancellation in a turbulent boundary layer, similar to that identified the case of interior turbulence in [14].

On the other hand, we find linearizing at a laminar solution that corresponding stability factors are large, which indicates that a laminar boundary layer cannot be modeled by slip even if the skin friction is small. This is also supported by the fact that laminar separation with no-slip is entirely different from (turbulent) separation with slip [16]. However, after laminar separation slightly viscous flow typically turns turbulent which can allow reattachment with a turbulent boundary layer.

Altogether, we find that slightly viscous flow in many cases can be modeled by slip/small friction boundary condition with a posteriori justification of the use of slip/small friction boundary conditions.

3 The Incompressible Navier-Stokes Equations

We consider the Navier-Stokes equations for an incompressible fluid of unit density with small viscosity $\nu > 0$ and small skin friction $\beta \geq 0$ filling a volume $\Omega$ in $\mathbb{R}^3$ surrounding a solid body with boundary $\Gamma$ over a time interval $I = [0, T]$; Find the velocity $u = (u_1, u_2, u_3)$ and pressure $p$ depending on $(x, t) \in \Omega \cup \Gamma \times I$, such that

\begin{align*}
\dot{u} + (u \cdot \nabla)u + \nabla p - \nabla \cdot \sigma &= f \quad &\text{in } \Omega \times I, \\
\nabla \cdot u &= 0 \quad &\text{in } \Omega \times I, \\
\left. u \right|_\Gamma &= g \quad &\text{on } \Gamma \times I, \\
\sigma_s &= \beta u_s \quad &\text{on } \Gamma \times I, \\
\left. u \right| (\cdot, 0) &= u^0 \quad &\text{in } \Omega,
\end{align*}

where $\dot{u} = \frac{\partial u}{\partial t}$, $u_n$ is the fluid velocity normal to $\Gamma$, $u_s$ is the tangential velocity, $\sigma = 2\nu \epsilon(u)$ is the stress with $\epsilon(u)$ the usual velocity strain, $\sigma_s$ is the tangential stress,
f is a given volume force, g is a given inflow/outflow velocity with \( g = 0 \) on a non-penetrable boundary, and \( u^0 \) is a given initial condition. We notice the skin friction boundary condition coupling the tangential stress \( \sigma_s \) to the tangential velocity \( u_s \) with

\[ \beta = \frac{U^2}{c_f}, \]

where \( c_f = \frac{2\tau}{\rho U^2} \) is the skin friction coefficient, with \( \beta = 0 \) for slip (and \( \beta >> 1 \) for no-slip).

Prandtl insisted on using a no-slip velocity boundary condition with \( u_s = 0 \) on \( \Gamma \), because his resolution of d’Alembert’s paradox hinged on discriminating potential flow by this condition. On the other hand, with our new resolution of d’Alembert’s paradox, relying instead on instability of potential flow, we are free to choose instead a friction force boundary condition, if data is available. Now, experiments show that the skin friction coefficient decreases with increasing Reynolds number \( Re \) as \( c_f \approx 0.05 \sim Re^{-0.2} \), so that \( c_f \approx 0.0005 \) for \( Re = 10^{10} \) and \( c_f \approx 0.005 \) for \( Re = 10^{5} \). Accordingly we model a turbulent boundary layer by friction boundary condition with a friction parameter \( \beta \approx 0.03 U Re^{-0.2} \).

We are now performing benchmark computations for tabulating values of \( \beta \) (or \( \sigma_s \)) for different values of \( Re \) by solving the Navier-Stokes equations with no-slip, and more generally for different values of \( \nu, U \) and length scale, since the dependence seems to be more complex than simply through the Reynolds number. Early results are reported in [14] with \( \sigma_s \approx 0.005 \) for \( \nu \approx 10^{-4} \) and \( U = 1 \), with corresponding velocity strain in the boundary layer \( 10^4 \sigma_s \approx 50 \) indicating that the smallest radius of curvature without separation in this case could be expected to be about 0.02.

We show in [14, 12, 15, ?, ?] that the Navier-Stokes equations (1) can be solved by a stabilized finite element referred to as G2 as an acronym for General Galerkin. G2 produces turbulent solutions characterized by substantial turbulent dissipation from the least squares stabilization acting as an automatic turbulence model, reflecting that the Navier-Stokes residual cannot be made small in turbulent regions. G2 has a posteriori error control based on duality and shows output uniqueness in mean-values such as lift and drag [14, 11, 10, 13, ?].

We find that G2 with slip is capable of modeling slightly viscous turbulent flow with \( Re > 10^6 \) of relevance in many applications in aero/hydro dynamics, including flying, sailing, boating and car racing, with hundred thousands of mesh points in simple geometry and millions in complex geometry, while according to state-of-the-art quadrillions is required [19]. This is because a friction-force/slip boundary condition can model a turbulent boundary layer, and interior turbulence does not have to be resolved to physical scales to capture mean-value outputs [14].

The idea of circumventing boundary layer resolution by relaxing no-slip boundary conditions introduced in [14], was used in [2] in the form of weak satisfaction of no-slip, which however misses the main point of using a force condition instead of a velocity condition.
4 The Linearized Dual Problem

We consider the following dual problem linearized between two turbulent flow velocities \( u \) and \( \bar{u} \) as in [14]: Given the (smooth) weight function \( \psi \), find \((\varphi, q)\) such that

\[
\begin{align*}
-\dot{\varphi} - (u \cdot \nabla) \varphi + \nabla \bar{u}^\top \varphi + \nabla q - \nu \Delta \varphi &= \psi \quad \text{in } \Omega \times I, \\
\nabla \cdot \varphi &= 0 \quad \text{in } \Omega \times I, \\
\varphi_n &= 0 \quad \text{on } \Gamma \times I, \\
\tau_s &= 0 \quad \text{on } \Gamma \times I, \\
\varphi(\cdot, T) &= 0 \quad \text{in } \Omega,
\end{align*}
\]

(2)

where \( \tau = 2\nu \epsilon(\varphi) \) and \( \tau_s \) the corresponding tangential stress, and \( \top \) denotes transpose. This is a linear convection-reaction-diffusion problem with the reaction coefficient \( \nabla \bar{u}^\top \) oscillating with large amplitude indicating local exponential instability.

5 Sensitivity Analysis

Let now \((\bar{u}, \bar{p})\) represent a perturbation of a base flow \((u, p)\) through a perturbation of the tangential stress \( \sigma_s \) into \( \bar{\sigma}_s \). By integrations by parts as in [14], using that both \((u, p)\) and \((\bar{u}, \bar{p})\) solve the same Navier-Stokes equations with somewhat different tangential stress, we obtain the following error representation formula

\[
((u - \bar{u}, \psi)) = <\sigma_s - \bar{\sigma}_s, \varphi_s>,
\]

(3)

where \(((\cdot, \cdot))\) and \(<\cdot, \cdot>\) denotes the appropriate \( L_2(\Omega \times I) \) and \( L_2(\Gamma) \) scalar product, respectively. Choosing now \( \bar{\sigma}_s = 0 \) corresponds to choosing a slip boundary condition, in which case the effect on the output \((u - \bar{u}, \psi))\) is given by \(<\sigma_s, \varphi_s>\). The corresponding stability factor defined by

\[
S = \frac{|<\sigma_s, \varphi_s>|}{\|\psi\|}
\]

(4)

where \(\|\psi\|\) is the \( L_2(\Omega \times I) \)-norm, then measures the output sensitivity of using a slip boundary condition.

6 Computational Results

We have computed the stability factor \( S \) for a set of test problems all giving small values, indicating that a slip boundary condition can be used for slightly viscous turbulent flow, as announced. The fact that the stability factor comes out small despite local exponential instability can be understood as an effect of cancellation as discussed in more detail in [14]. We also show that linearization at a laminar solution gives large stability factors reflecting lack of cancellation, which shows that a laminar boundary layer cannot be modeled by slip.
References


