Computation of the Optimal Velocity Disturbances of the Low Reynolds Number Flow Past a Circular Cylinder using a Stabilized Finite Element Method

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Computation of the Optimal Velocity Disturbances of the Low Reynolds Number Flow Past a Circular Cylinder using a Stabilized Finite Element Method

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ABSTRACT

The stability of the cylinder wake flow is investigated in a global sense. For sub-critical Reynolds numbers the cylinder wake flow is two dimensional and stable. Disturbing the wake leads to an amplification behavior in terms of the disturbance global energy. A steady base flow is generated, by solving the Navier-Stokes equations. The base flow is disturbed arbitrarily, and the optimal initial disturbances leading to maximum energy amplification, at certain instances in time, are computed through a primal-dual optimization for different low Reynolds numbers. Governing equations are solved using a stabilized finite element method, and implemented in the free software FEniCS (www.fenics.org). More attention should be paid to the mesh to improve the accuracy, and using adaptive mesh is particularly recommended.

Beräkning av optimala störningar för flödet kring en circlär cylinder vid låga Reynolds tal med en stabiliserad finit elementmetod

Sammanfattning

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CHAPTER 1 INTRODUCTION

Incompressible flow past a circular cylinder is a classical problem of great interest for its importance as a prototype for bluff-body wake flows. The flow is characterized by a single non-dimensional parameter, the Reynolds number \( \text{Re} = U_\infty D/\nu \), where \( \nu \) is the kinematic viscosity, \( U_\infty \) is the free-stream velocity and \( D \) is the cylinder diameter. For values of \( \text{Re} \) below a critical value \( \text{Re}_c \approx 46 \) [1], the flow is globally stable and a steady two-dimensional wake appears behind the cylinder. For \( \text{Re} > \text{Re}_c \), the flow becomes unstable through a Hopf bifurcation leading to the oscillating Karman vortex street. However, for \( \text{Re} < \text{Re}_c \), the flow can still support the transient amplification of disturbances in the wake. This transient behavior at sub-critical Reynolds numbers will be the object of this work.

The classical approach to investigate flow stability, valid for nearly parallel flows, is the local approach, in which periodic waves are assumed to evolve in a flow which is homogenous in the streamwise direction. The basic flow for stability analysis is that obtained by extending the velocity profile at one streamwise location indefinitely. The proper, but much more expensive, approach is the global approach, in which the flow dynamics is viewed as a result of the interactions between global modes existing in the entire physical domain. In the local approximation, there are two types of instabilities: absolute and convective. In an absolutely unstable flow, any initial disturbance will grow in time and evolve into a self-sustained oscillation. Local absolute instability has been shown to be necessary for global instability. In a convectively unstable flow, unstable disturbances will be amplified and carried away by the mean flow, leaving finally the flow undisturbed. In a global setting, local convective instability is giving a transient amplification of disturbances which will all eventually decay. [2][3][4]

The purpose of this work is to find the optimal disturbances giving the maximum transient amplification in time in a global sense, for the incompressible flow around a circular cylinder. Our approach is based on a primal-dual optimization algorithm with the global kinetic energy as instability measure. A similar approach has been used in [5] to find optimal disturbances amplified in the streamwise direction for a boundary layer flow. First the incompressible Navier-Stokes equations (NSE) are solved to get a steady base flow. The equations are then linearized around the base flow to solve for small initial disturbances. To complete the optimization procedure, the corresponding dual equations are also derived and solved.

To solve the governing equations numerically, a finite element method is used. A standard Galerkin finite element method (FEM) has proven to be very successful in solving many kinds of partial differential equations. However, applying FEM to fluid dynamics and other transport problems, serious numerical instabilities are encountered. In order to overcome the drawbacks of FEM for these problems, stabilized FEM was introduced. The Galerkin least squares finite element method (GLS) was initially developed by Hughes, Tezduyar, and Johnson, with co-workers. It consists of FEM with
a weighted least squares stabilization of the residual, leading to improved numerical stability properties with minimal loss of accuracy [6]. We here refer to GLS as a General Galerkin (G2) method. G2 is implemented in FEniCS, a free open source software, to solve the NSE. To solve the linearised Navier-Stokes (LNSE) and its dual (DNSE), G2 is used, and implemented in FEniCS within the scope of this work.
CHAPTER 2 PROBLEM FORMULATION

An initial disturbance is introduced into a steady cylinder wake flow and its amplification is maximized at some time instance through a primal-dual (adjoint based) optimization. We first present the three sets of Navier-Stokes equations involved, along with their appropriate boundary conditions. The primal-dual optimization is described later in this chapter.

2.1 The Navier-Stokes equations

The problem is studied for two-dimensional flow over an infinitely long circular cylinder. A Cartesian coordinate system is placed in the center of the cylinder, with the \( x - axis \) pointing in the flow direction and the \( z - axis \) running along the cylinder center line. For \( Re \leq 190 \), the flow can be described by the two-dimensional unsteady Navier-Stokes equations (NSE)

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \frac{1}{Re} \Delta u = f, \\
\nabla \cdot u = 0,
\]

where \( u(x,t) \) is the velocity vector, \( p(x,t) \) is the pressure and \( f(x,t) \) is the external force term. The equations (2.1) are made dimension-less using the cylinder diameter \( D \) as the characteristic length scale, the velocity of incoming uniform velocity \( U_\infty \) as the reference velocity and \( \rho U_\infty^2 \) as reference pressure. On the surface of the cylinder, the no-slip and no-penetration conditions require both velocity components to vanish. Far away from the cylinder, the flow reaches the incoming uniform stream. Therefore, we have the following set of boundary conditions:

\[
u_1 = u_2 = 0 \quad \text{on} \quad \Gamma_c, \\
u_1 \to 1 \quad \text{as} \quad x^2 + y^2 \to \infty, \\
u_2 \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty, \tag{2.2}\]

The total velocity field can be decomposed into the sum a steady part and a small unsteady disturbance as

\[
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \\ P \end{bmatrix} + \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{p} \end{bmatrix} = \mathbf{U} + \tilde{\mathbf{u}}, \tag{2.3}\]

where \( \mathbf{U} \) is the mean flow governed by the steady version of (2.1), and \( \tilde{\mathbf{u}} \) is the disturbed field described by the linearised Navier-Stokes equations (LNSE). Introducing (2.3) into (2.1) and linearising we obtain the LNSE (dropping tildes)
On the cylinder surface the no-slip and no-penetration conditions still apply, and we assume that the disturbances die far enough from the cylinder. Therefore, the boundary conditions for (2.4) are:

\[
\begin{align*}
\hat{u}_1 &= \hat{u}_2 = 0 \quad \text{on} \quad \Gamma_c, \\
\hat{u}_1, \hat{u}_2, p &\to 0 \quad \text{as} \quad x^2 + y^2 \to \infty.
\end{align*}
\]

To derive the corresponding dual problem, introduce \( \hat{\mathbf{u}} = [\hat{u}_1 \quad \hat{u}_2 \quad \hat{p}]^T \), where \( \hat{u}_1 \) and \( \hat{u}_2 \) are the dual velocity components and \( \hat{p} \) is the dual pressure. Then we consider the inner product between the dual vector and the residual of (2.4), and integrate by parts over time and space to move the derivatives from \( \mathbf{u} \) to \( \hat{\mathbf{u}} \). The obtained dual problem (DNSE) is

\[
\begin{align*}
-\frac{\partial \hat{u}}{\partial t} + \nabla U \cdot \hat{u} - U \cdot \nabla \hat{u} + \nabla \hat{p} - \frac{1}{\text{Re}} \Delta \hat{u} &= f, \\
\nabla \cdot \hat{u} &= 0.
\end{align*}
\]

Note that we have a minus sine for time derivative, which implies backward integration in time. Integration by parts gives rise to integrations over the boundary from which we get the boundary conditions for (2.6) to be

\[
\begin{align*}
\hat{u}_1 &= \hat{u}_2 = 0 \quad \text{on} \quad \Gamma_c, \\
\hat{u}_1, \hat{u}_2, \hat{p} &\to 0 \quad \text{as} \quad x^2 + y^2 \to \infty.
\end{align*}
\]

Detailed derivations of the LNSE and DNSE are provided in appendix A.

2.2 Primal-dual optimization

To investigate the global amplification behavior of wake flows, the kinetic energy of the disturbances at a specific time instance is defined by

\[
E(t) = \int_\Omega |u(t)|^2 d\Omega = (u, u).
\]

Moreover, we define the disturbance growth as
(2.9) \[ G(t_0, T) = \frac{E(T)}{E(t_0)}, \]

where \( E(t_0) \) and \( E(T) \) are the energies of initial and final disturbance fields of (2.4).

Let \( q = u(0) \) be the initial disturbance velocity and \( u = u(T) \) the disturbance velocity resulting from solving the LNSE. The disturbance growth can be defined as

(2.10) \[ G = \left( \frac{u}{u} \right) \left( \frac{q}{q} \right), \]

Furthermore, let \( A \) be the linear operator representing the integration of (2.4) with (2.5) from \( t = 0 \) to \( t = T \), and \( A^* \) the corresponding dual operator representing the integration of (2.6) with (2.7) from \( t = T \) to \( t = 0 \). Since \( A \) is a linear homogenous operator, we can consider the input-output formulation

(2.11) \[ u = Aq. \]

The maximum growth may then be written as

(2.12) \[ G_{\text{max}} = \frac{(A^* A, q)}{(q, q)}, \]

where \( q \) is the eigenvector corresponding to the largest eigenvalue of the eigenvalue problem

(2.13) \[ A^* Aq = \lambda q. \]

Then \( G_{\text{max}} \) is the maximum positive real eigenvalue \( \lambda_{\text{max}} \) of (2.13). One way to calculate the optimal initial disturbance and its associated maximum growth is by power iterations

(2.14) \[ q^{n+1} = A^* Aq^n, \]

where the initial disturbance is scaled to the given initial energy in each iteration.
CHAPTER 3 NUMERICAL APPROACH

We now present the space-time Galerkin least squares finite element method (G2) used to solve the three Navier-Stokes problems at hand; NSE, LNSE and DNSE. In [7], G2 is extended and applied to NSE as a general method for laminar and turbulent flow in the form of an adaptive method with automatic choice of the mesh in space-time based on a posteriori error estimation by solving the associated dual problem DNSE. First, we formulate the variational form of NSE using one form of G2, and extend it to LNSE and DNSE. Then, the implementation with FEniCS is described before we highlight the optimization algorithm.

3.1 G2 for the Navier-Stokes equations

Recall NSE defined on a two-dimensional finite domain $\Omega$, with boundary $\Gamma$, and a time interval $I$.

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \frac{1}{\text{Re}} \Delta u = f \quad \text{in} \quad \Omega \times I,
\]

\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times I,
\]

\[
u = g \quad \text{on} \quad \Gamma,
\]

\[
u(.,0) = u^0 \quad \text{in} \quad \Omega,
\]

where $u(x,t)$ is the velocity, $p(x,t)$ the pressure, Re the Reynolds number, $f$ the external force, $g$ the Dirichlet boundary data, and $u^0$ the initial data.

We discretize NSE using G2 in the form of the cG(1)cG(1) method. With cG(1) in space, both trial and test functions are continuous piecewise linear. With cG(1) in time, the trial functions are continuous piecewise linear and the test functions are piecewise constant. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be a sequence of discrete time steps with associated time intervals $I_n = (t_{n-1}, t_n)$ of length $k_n = t_n - t_{n-1}$ and space-time slabs $S = \Omega \times I_n$, and let $W^n$ be a finite element space consisting of continuous piecewise linear functions on a mesh $\mathcal{K}$ of mesh size $h(x)$ with $W_g^n$ the functions in $W^n$ satisfying the Dirichlet boundary condition on $\Gamma$. We seek functions $(u_h^n, p_h^n)$, continuous piecewise linear in space and time, and the cG(1)cG(1) method for NSE with homogenous Dirichlet boundary conditions, reads as follows:

For $n = 1, \ldots, N$, find $u = (u_h^n, p_h^n) \equiv (u_h(t_n), p_h(t_n))$ with $u_h^n \in V_0^n \equiv [W_0^n]^3$ and $p_h^n \in Q_0^n \equiv W^n$, such that

\[
(R(u), v)_n + SD_\delta (u; v)_n = 0
\]

for all $v = (v, q) \in V_0 \times Q$, where $\langle \cdot, \cdot \rangle$ denotes an inner product, with Galerkin term.
\[(R(u), v)_n = \left( (u^n_h - u^{n-1}_h) k^{-1} + \bar{u}^n_h \nabla \bar{u}^n_h - f, v \right) + \left( \text{Re}^{-1} \nabla \bar{u}^n_h, \nabla v \right) - \left( p^n_h, \nabla \cdot v \right) + \left( \nabla \cdot \bar{u}^n_h, q \right), \]

where \( \bar{u}_h^n \equiv \frac{1}{2}(u^n_h - u^{n-1}_h) \), and the stabilizing weighted least squares term

\[(SD_0(u; v)_n = \left( \delta_1 \left( \bar{u}^n_h \nabla \bar{u}^n_h + \nabla p^n_h - f \right), \bar{u}^n_h \nabla v + \nabla q \right) + \left( \delta_2 \nabla \cdot \bar{u}^n_h, \nabla \cdot v \right). \]

This method corresponds to a second order accurate Crank-Nicolson time-stepping. The time derivative disappears in the \( SD_0 \)-term because the test functions are piecewise constant in time. \( \delta_1 \) and \( \delta_2 \) are defined as

\[
\delta_1 = \begin{cases} 
\frac{1}{2} \kappa_1 \left( k^{-1} + |u|^2 h^{-2} \right) & \text{if } \text{Re}^{-1} < u^n_h, \\
\kappa_1 h^2 & \text{otherwise},
\end{cases}
\]

\[
\delta_2 = \begin{cases} 
\kappa_2 h & \text{if } \text{Re}^{-1} < u^n_h, \\
\kappa_2 h^2 & \text{otherwise},
\end{cases}
\]

with \( \kappa_1 \) and \( \kappa_2 \) positive constants of unit size.

### 3.2 G2 for LNSE and DNSE

We now formulate G2 for LNSE. The Galerkin term for LNSE is defined as

\[(R(u), v)_n = \left( (u^n_h - u^{n-1}_h) k^{-1} + U^n_h \nabla \bar{u}^n_h + \bar{u}^n_h \nabla U - f, v \right) + \left( \text{Re}^{-1} \nabla \bar{u}^n_h, \nabla v \right) - \left( p^n_h, \nabla \cdot v \right) + \left( \nabla \cdot \bar{u}^n_h, q \right), \]

where \( u = (u^n_h, p^n_h) \) is the approximation of the disturbed field and \( U \) is the steady base velocity. We also define the stabilizing term as

\[(SD_0(u; v)_n = \left( \delta_1 \left( U \nabla \bar{u}^n_h + \bar{u}^n_h \nabla U + \nabla p^n_h - f \right), U \nabla v + v \nabla U + \nabla q \right) + \left( \delta_2 \nabla \cdot \bar{u}^n_h, \nabla \cdot v \right), \]

with the same \( \delta_1 \) and \( \delta_2 \) as in (3.5).

For DNSE, G2 is formulated with inverse time stepping, resulting in negative time steps \( k_n \). The Galerkin term for DNSE is

\[(R(u), v)_n = \left( -(u^n_h - u^{n-1}_h) k^{-1} - U^n_h \nabla \bar{u}^n_h + \nabla U \cdot \bar{u}^n_h - f, v \right) + \left( \text{Re}^{-1} \nabla \bar{u}^n_h, \nabla v \right) - \left( p^n_h, \nabla \cdot v \right) + \left( \nabla \cdot \bar{u}^n_h, q \right), \]
where \( \mathbf{u} = (u_h^n, p_h^n) \) is the approximation of the dual-disturbance field and \( U \) is the steady base velocity. The stabilizing term for DNSE is

\[
SD_0 (\mathbf{u}; \mathbf{v}) \equiv \left( \delta_1 \left( -U \cdot \nabla \overline{u}_h^n + \nabla U \cdot \overline{u}_h^n + \nabla p_h^n - f \right) - U \cdot \nabla v + \nabla U \cdot v + \nabla q \right) \\
+ \left( \delta_2 \nabla \cdot \overline{u}_h^n, \nabla \cdot v \right)
\]

with the same \( \delta_1 \) and \( \delta_2 \) as in (3.5).

3.3 Numerical boundary conditions

Since we are solving NSE, LNSE and DNSE numerically on a bounded domain \( \Omega = [x_1, x_2] \times [y_1, y_2] \), we need to modify the associated boundary conditions (2.2), (2.5) and (2.7). Figure 1 shows a sketch of \( \Omega \), where the positive \( x \)-axis points to the right and positive \( y \)-axis points upwards.

At the cylinder surface, all velocities vanish due to no-slip and no-penetration conditions. For NSE, \( u_1 = 1 \) and \( u_0 = 0 \) at the inflow, \( \Gamma_{left} \), and \( u_2 = 0 \) at the upper and lower boundaries. Pressure is taken to be zero at the outflow, \( \Gamma_{right} \), since the domain is considered to be long enough. For LNSE and DNSE all velocities as well as pressure are taken to be zero all over the boundary.

![Figure 1](image)

Figure 1 Sketch of the finite domain

3.4 Implementation with FEniCS

FEniCS is a free software for the Automation of Computational Mathematical Modeling (ACMM) developed by research groups at a number of technological universities and institutes. In this work we use three components of FEniCS: FIAT, FFC and DOLFIN. FIAT generates finite elements of different types. FFC is a variational form compiler that generates a C++ header file for the assembly of matrices and vectors. DOLFIN is the C++ interface of FEniCS, which provides many tools for assembling and solving the linear systems. DOLFIN provides solvers implemented for a number of well-known problems. Results can be saved selectively to be visualized with different visualization
Among the implemented solvers in DOLFIN is the NSE solver. It uses the fixed point iteration method to solve the nonlinear system of algebraic equations at each time step, which converges under CFL condition $\frac{\Delta t}{h} < 1$. As a part of this work, solvers for LNSE and DNSE are implemented in DOLFIN. Variational forms were compiled with FFC, which uses FIAT to construct the finite elements. Then, the resulting header files were used in DOLFIN to assemble the linear systems. FEniCS currently supports triangle (2D) and tetrahedron meshes (3D). Our investigation considers 2D problem, but the solver works also for 3D.

### 3.5 Primal-dual optimization algorithm

The primal-dual optimization algorithm converges in three to four iterations, provided that a dominant eigenvalue of the eigenvalue problem (2.13) exists. The optimization problem is solved using the power iteration method (2.14), and includes the following steps:

1. Solve the LNSE with initial arbitrary disturbance $u^0 = u(t_0)$, after normalizing the velocity with the square root of the global energy $E(t_0)$.
2. Assign the final velocity from LNSE, $\hat{u}(T) = u(T)$, as an initial condition to the DNSE and solve backward to $t_0$.
3. Assign the final velocity from DNSE, $u(t_0) = \hat{u}(t_0)$, as an initial condition to the LNSE and solve forward to $T$ after normalizing as in step 1. If $\left| \frac{G^{n+1} - G^n}{G^{n+1}} \right|$ falls below a small prescribed tolerance, the algorithm terminates. Otherwise, return to step 2.
CHAPTER 4 RESULTS AND DISCUSSION

We run the optimization algorithm for a mesh generated by PDE-ToolBox in MATLAB over the domain \([-25:50] \times [-20:20]\), where the cylinder of unit diameter is placed in the origin \((0,0)\). The mesh, in Figure 2, consists of 7,360 nodes and 14,464 triangles, and it is fine around the cylinder and getting coarse as we move away from it. We investigate for low Re numbers, 20, 30 and 40, and optimize for different times, 5, 10, 15 and 20.

![Figure 2](image)

Figure 2 A 7,360 nodes and 14,464 triangles mesh on a \([-25:50] \times [-20:20]\) domain

4.1 Validation of FEniCS NSE solver

First, we use the NSE solver already implemented in FEniCS to generate the base flow for the different Re. To validate the code, we compare with others work. In [9], a list of wake length, measured from the rear stagnation point, is provided from different studies, for \(Re = 20\) and 40. Table 1 shows the agreement between our wake length results and the list from [9].

<table>
<thead>
<tr>
<th>Study</th>
<th>Re = 20</th>
<th>Re = 40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dennis &amp; Chang (1970)</td>
<td>0.94</td>
<td>2.35</td>
</tr>
<tr>
<td>Coutanceau &amp; Bouard (1977)</td>
<td>0.73</td>
<td>1.89</td>
</tr>
<tr>
<td>Fornberg (1980)</td>
<td>0.91</td>
<td>2.24</td>
</tr>
<tr>
<td>Ye et al (1999)</td>
<td>0.92</td>
<td>2.27</td>
</tr>
<tr>
<td>Giannetti and Luchini (2004)</td>
<td>0.92</td>
<td>2.24</td>
</tr>
<tr>
<td>Current</td>
<td>0.92</td>
<td>2.23</td>
</tr>
</tbody>
</table>

Table 1 Wake length for \(Re = 20\) and \(Re = 40\) from different studies
4.2 Energy evolution in time

To verify the amplification behavior of the current flow, we integrate the LNSE forward in time up to $T = 40$. Figures 3, 4 and 5 show the evolution in time of the disturbance global energy for the three Re. The initial disturbance is chosen to be a localized impulse in the near wake flow, close to the cylinder surface, based on receptivity results from [9]. The initial condition is defined in terms of a stream function

$$(4.1) \quad \psi (x, y) = A \exp \left( - \frac{x^2}{\bar{x}^2} - \frac{y^2}{\bar{y}^2} \right),$$

where $A$ is the amplitude, $\bar{x} = (x - x_0)/l_x$ and $\bar{y} = (y - y_0)/l_y$. $(x_0, y_0)$ and $(l_x, l_y)$ are the impulse location and length, respectively [10]. It is easy to verify that (4.1) satisfies the incompressibility condition $\nabla \cdot u = 0$. We have chosen $(x_0, y_0) = (0.8, 0.3)$, while $A$ and $(l_x, l_y)$ are chosen arbitrarily.

At the beginning, the energy decades rapidly, before it starts to amplify and decay again. Clearly, as Re increases, amplification starts earlier and reaches its peak later and the amplification value at the peak increases as well. Next step is to start the optimization algorithm, which will result in the maximum amplifications, at the given times, and their associated initial disturbances.

![Figure 3](energy_growth.png)

*Figure 3* Energy growth of the initial disturbance for Re = 20
**Figure 4** Energy growth of the initial disturbance for Re = 30

**Figure 5** Energy growth of the initial disturbance for Re = 40
4.3 Optimization results

The power iteration (2.14) is used to find the maximum energy growth $G_{\text{max}}$ and the associated optimal initial disturbance $q$. The initial guess is taken to be $q^0 = (\psi_y, -\psi_x)$, where subscripts denote partial derivatives. We apply the optimization algorithm described in Chapter 2, with a prescribed tolerance $TOL = 0.1$. Power iteration converged in two to four iterations.

$G_{\text{max}}$ at different times, in figures 6, 7 and 8, represent an upper bound of disturbance amplification. However, $G_{\text{max}}$ for $Re = 30$ at $t = 15$ looks odd, since it is less than $G_{\text{max}}$ at $t = 10$ and $t = 20$. Further investigation for more times $T$ and probably finer meshes can improve the accuracy of the computed maximum amplifications.

4.4 Mesh refinement effect

Finer mesh, which consists of 22,674 nodes and 45,040 triangles, is generated using the PDE-ToolBox in MATLAB to have an idea about the mesh refinement effect. Figure 9 shows the evolution in time of the global energy for $Re = 40$ and $T = 5$, from the last iteration. The relative error $(G_{\text{fine}} - G_{\text{coarse}})/G_{\text{fine}}$ is found to be less than the prescribed tolerance. For $T = 20$, figure 10 clearly shows a big error in the maximum amplification.

As Reynolds number and the time we optimize at increase, the disturbances move away from the cylinder. The energy integration will not be well approximated on the coarse mesh comparing to the fine one.

To have more accurate results, we can refine the whole domain. But the computation cost then will increase, not only due to the size of the linear systems assembled and solved, but also because of the restriction on the time-step satisfying the CFL condition. Alternatively, we can refine certain parts of the domain where the disturbances reach, and leave the rest of the domain relatively coarse. Another option is to start with a very coarse mesh and refine as we solve.
Figure 6 Energy growth of the optimal disturbances for Re = 20

Figure 7 Energy growth of the optimal disturbances for Re = 30
Figure 8 Energy growth of the optimal disturbances for $Re = 40$

Figure 9 Energy growth using coarse and fine meshes, for $Re = 40$ and $T = 5$
Figure 10 growth using coarse and fine meshes, for Re = 40 and T = 20
CHAPTER 5 CONCLUSION

In this work, the stability of a steady two-dimensional incompressible flow over a circular cylinder is investigated, by introducing a disturbance in the cylinder wake. The governing equations are solved numerically using a stabilized finite element method. The incompressible Navier-Stokes equations solver already implemented in FEniCS is used to generate the base flow, and it is validated through comparison with results from other studies. Solvers for the linearised Navier-Stokes equations and its dual are implemented in FEniCS within this work.

Stability investigation is done globally with the kinetic energy as a stability measure. Solving the linearised Navier-Stokes equations for three Reynolds numbers, the flow has shown amplification behavior as expected. Primal-dual optimization algorithm, based on power iteration, is performed to find the maximum growth at certain instance in time, and it associated optimal disturbance.

One finer mesh, than the one used in the analysis, was used to investigate the accuracy of one of the results. We found the error increasing as Reynolds number and time increase. Further investigation on the mesh shall be performed for different Reynolds numbers and time instances. Selectively or adaptively refined meshes can also be considered.

The involved solvers can be used, or easily improved, for further investigation. In particular, other Reynolds numbers and times can be investigated. It is also possible to investigate the stability of three-dimensional flows.
APPENDIX A

We here present the derivation of the linearized Navier-Stokes problem and its corresponding dual.

A.1 Linearized Navier-Stokes equations

Recall the incompressible Navier-Stokes equations (2.1) in Cartesian tensor notation

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_i \partial x_j} = 0,
\]

(A.1)

\[
\frac{\partial u_i}{\partial x_i} = 0.
\]

where the external force term is dropped for simplicity. Then, we decompose the flow into a steady part and a small disturbance:

\[
\begin{align*}
\tilde{u}_i &= U_i + \tilde{u}_i, \\
p &= P + \tilde{p}.
\end{align*}
\]

(A.2)

Now, we put (A.2) in (A.1) to get

\[
\begin{align*}
\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial U_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial \tilde{u}_i}{\partial x_j} + U_j \frac{\partial U_i}{\partial x_j} + \frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial P}{\partial x_i} \\
&\quad - \frac{1}{Re} \frac{\partial^2 \tilde{u}_i}{\partial x_i \partial x_j} - \frac{1}{Re} \frac{\partial^2 U_i}{\partial x_i \partial x_j} = 0,
\end{align*}
\]

(A.3)

\[
\frac{\partial \tilde{u}_i}{\partial x_i} + \frac{\partial U_i}{\partial x_j} = 0.
\]

The steady terms will vanish since they satisfy (A.1) and the small term is ignored because it is the square of the small disturbance. The linearized problem, after dropping tildes, is

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_i \partial x_j} = 0,
\end{align*}
\]

(A.4)

\[
\frac{\partial u_i}{\partial x_i} = 0,
\]

together with boundary conditions (2.5).
A.2 Dual equations corresponding to the LNSE

To derive the dual of (A.4), we introduce the dual velocity and pressure variables vector \( (\mathbf{\varphi}_i, \theta) \). We use different notation than in Chapter 2 to clearly distinguish the variables. Then, we consider the inner product between the dual vector and the residual of (A.4) and integrate over space and time

\[
\int_0^T \left( \int_\Omega \left( \begin{array}{c} \frac{\partial u_i}{\partial t} + u_j \frac{\partial U_i}{\partial x_j} + U_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j^2} \\ \frac{\partial u_i}{\partial x_j} \end{array} \right) \right) d\Omega dt = 0.
\]

We use integration by parts to move the derivatives form the primal to the dual variables.

We consider term by term before we sum everything up.

Term 1
\[
\int_0^T \int_\Omega \frac{\partial u_i}{\partial t} \varphi_i d\Omega dt = \int_\Omega \left( \left. u_i \varphi_i \right|_t - \left. \frac{\partial \varphi_i}{\partial t} \right|_0 \right) d\Omega
\]

Term 2
\[
\int_0^T \int_\Omega u_j \frac{\partial U_i}{\partial x_j} \varphi_i d\Omega dt = \int_\Omega \left( \left. u_j \frac{\partial U_i}{\partial x_j} \right|_t \right) d\Omega
\]

Term 3
\[
\int_0^T \int_\Omega U_j \frac{\partial u_i}{\partial x_j} \varphi_i d\Omega dt = \int_\Omega \left( \left. U_j u_i \varphi_i \right|_t - \left. U_j \frac{\partial \varphi_i}{\partial x_j} \right|_0 \right) d\Omega dt
\]

Term 4
\[
\int_0^T \int_\Omega \frac{\partial p}{\partial x_i} \varphi_i d\Omega dt = \int_\Omega \left( \left. p \varphi_i \right|_t - \left. \frac{\partial \varphi_i}{\partial x_i} \right|_0 \right) d\Omega dt
\]

Term 5
\[
\int_0^T \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j} \varphi_i d\Omega dt = -\frac{1}{Re} \int_0^T \left( \left. \frac{\partial u_i}{\partial x_j} \varphi_i \right|_t - \left. \frac{\partial u_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} d\Omega \right) dt
\]

\[
= -\frac{1}{Re} \int_0^T \left( \left. \frac{\partial u_i}{\partial x_j} \varphi_i \right|_t - \left. u_i \frac{\partial \varphi_i}{\partial x_i} \right|_t + \left. \frac{\partial \varphi_i^2}{\partial x_j \partial x_j} d\Omega \right) dt
\]

Term 6
\[ \int_0^\tau \int_\Omega \frac{\partial u_i}{\partial x_i} \theta \, d\Omega \, dt = \int_0^\tau \left[ u_i \theta \big|_\Gamma - \int_\Omega u_i \frac{\partial \theta}{\partial x_i} \, d\Omega \right] \, dt \]

All space boundary terms vanish due to (2.5), and \( u_i \theta \big|_\Gamma = u_i (T) \theta (T) - u_i (0) \theta (0) = 0 \).

Summing up we reach to

\[ \frac{\partial \phi_i}{\partial t} + \phi_j \frac{\partial U_j}{\partial x_i} - U_j \frac{\partial \phi_i}{\partial x_j} + \frac{\partial \theta}{\partial x_i} = \frac{1}{\text{Re}} \frac{\partial^2 \phi_i}{\partial x_j \partial x_j}, \]

(A.6) \[ \frac{\partial \phi_i}{\partial x_i} = 0. \]
REFERENCES


[8] www.fenics.org

