Modelling the Price of a Credit Default Swap

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Abstract

This project describes credit default swap (CDS) and shows how to calculate a fair value for such a contract. The stochastic evaluation of both the interest rate and the default intensity are first studied independent by using one factor a Vasicek model for a bond and a one factor model for the probability of no default. After validations the combined effect of stochastic interest rates and default intensities is examined to calculate a more accurate value with a two factor model for the CDS.

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Modell av priset för en Credit Default Swap

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1 Introduction

In recent years credit derivatives enjoyed one of the largest growth of all the markets. Their purpose is to manage and trade credit risk, i.e. the risk that a borrower may not be able to pay back a loan in time.

A credit default swap (CDS) is the most common and straightforward type: it is a contract that provides insurance against the risk of default by a particular company. The company is known as the reference entity and a default by the company is known as a credit event. The credit event that triggers a default is defined in the contract: it usually includes bankruptcy, restructuring and failure to pay.

The buyer of the insurance has the right to sell a bond issued by the company in the case of a credit event. The buyer may deliver this bond at par value or receive the par value minus a recovery rate. How the swap is settled is defined in the contract. In exchange of the right granted by the CDS, the buyer agrees to make fixed regular payments to the seller during the lifetime of the contract or until a credit event occurs. Thus, the CDS consists of two cash flow legs: the fixed leg and the default leg (figure 1).

The price of the credit default swap is described by the difference between the cash flow in both legs. Assume that the par value is normalized to unity and in the credit event the buyer receives the par value minus recovery rate $R$. Define

- $T = t_n$: The life time of the credit default swap
- $Q(t)$: The probability of having no default
- $K$: The fixed annual insurance payment to compensate for the risk of default
- $R$: The recovery rate reducing the payment in the case of credit event
- $P(t, T)$: The discount function for a zero bond at time $t$ maturing at $T$
- $C(s, t)$: The accrual function denoting the fraction of year between date $s$ and $t$
- $V_{fixed}$: The present value of the fixed leg
- $V_{default}$: The present value of the default leg
- $V_{CDS}$: The present value of the credit default swap

The fixed leg is obtained from the present value of all the payments made by the protection buyer,

$$V_{fixed} = K \sum_{i=1}^{n} P(0, t_i)Q(t_i)C(t_{i-1}, t_i)$$  \hspace{1cm} (1)

The default leg is given by expected amount received by the protection buyer in the case of a credit event

$$V_{default} = \int_0^T (1 - R)P(0, t)(1 - Q(t))dt$$  \hspace{1cm} (2)

The credit default swap is simply the difference between these two legs

$$V_{CDS} = \int_0^T (1 - R)P(0, t)(1 - Q(t))dt - K \sum_{i=1}^{n} P(0, t_i)Q(t_i)C(t_{i-1}, t_i)$$  \hspace{1cm} (3)

The values of $P(0, t_i)$, $Q(t_i)$ will be calculated using Monte Carlo methods to simulate possible realizations of the interest rate and the default intensity. All other values are assumed to be known.
2 The Risk Free Interest Rate

2.1 Modelling the discount function

Assume that interest rate process is described by a mean reversion forecast model, i.e.

\[ dr_t = a(b - r_t)dt + \sigma dz_t \]  

where \( r \) is an instantaneous short rate, \( a(b - r) \) is a mean reversion drift, \( \sigma \) is a volatility, \( dz \) is a Wiener process and \( a, b, \sigma \) are constants. The short rate is pulled to a level \( b \) at rate \( a \). Superimposed upon this “pull” is a normally distributed stochastic term \( \sigma dz \). It is easy to obtain the expected value of the discount function in time \( t \)

\[ P(t, T) = E \left[ e^{-\int_t^T r_s ds} \right] \]  

Let us convert a continuous-time model (eq.4) to discrete-time. Applying rules from conventional differential calculus to \( e^{r_t} \) give rise to:

\[ de^{at}r_t = ae^{at}r_t dt + e^{at}r_t dr_t \]

\[ = ae^{at}r_t dt + e^{at}r_t [a(b - r_t)dt + \sigma dz_t] \]

\[ = abe^{at} dt + \sigma e^{at} dz_t \]

This implies that

\[ e^{as}r_s - e^{at}r_t = ab \int_t^s e^{a\tau} d\tau + \sigma \int_t^s e^{a\tau} dz_{\tau} \]  

\[ = b \left( e^{as} - e^{at} \right) + \sigma \int_t^s e^{a\tau} dz_{\tau} \]  

and

\[ r_s = r_t e^{-a(s-t)} + b \left( 1 - e^{-a(s-t)} \right) + \sigma \int_t^s e^{-a(s-\tau)} d\tau \]

Integrate over the lifetime of the bond bond \([t, T]\) to determine possible realizations for the accumulated interest payed a long one single random path

\[ \int_t^T r_s ds = r_t \int_t^T e^{-a(s-t)} ds + b \int_t^T \left( 1 - e^{-a(s-t)} \right) ds + \sigma \int_t^T \int_t^s e^{-a(s-\tau)} d\tau ds \]  

\[ \int_t^T r_s ds = r_t \frac{1}{a} \left( 1 - e^{-a(T-t)} \right) + b(T-t) - \frac{b}{a} \left( 1 - e^{-a(T-t)} \right) + \sigma \frac{1}{a} \int_t^T \left( 1 - e^{-a(T-\tau)} \right) d\tau \]

where

\[ \frac{\sigma}{a} \int_t^T (1 - e^{-a(T-\tau)}) d\tau \]

has a normal distribution with zero mean \( N \left( 0, \frac{\sigma^2}{a^2} \int_t^T (1 - e^{-a(T-\tau)})^2 d\tau \right) \) and \( a \neq 0 \).
The value of discount function is obtained from the expected value of all the possible realizations

\[ P(t, T) = E \left[ \exp \left( -r_t \frac{1}{a} \left( 1 - e^{-a(T-t)} \right) - b(T - t) + \frac{b}{a} \left( 1 - e^{-a(T-t)} \right) - \sigma \int_t^T \left( 1 - e^{-a(T-\tau)} \right) d\tau \right) \right] \]  

(15)

where

\[ \frac{\sigma}{a} \int_t^T \left( 1 - e^{-a(T-\tau)} \right) d\tau \]  

has a normal distribution

\[ N \left( 0, \frac{\sigma^2}{2a^3} \left( -3 + e^{-2a(T-t)} \left( -1 + 4e^{a(T-t)} + 2ae^{2a(T-t)}(T - t) \right) \right) \right) \]  

(17)

assuming \( a \neq 0 \). The limit \( a \to 0 \) reduces these expression further to

\[ P(t, T) = E \left[ \exp \left( -r_t(T - t) - \sigma \int_t^T (T - \tau)d\tau \right) \right] \]  

(18)

where

\[ \sigma \int_t^T (T - \tau)d\tau \]  

has a normal distribution

\[ N \left( 0, \frac{\sigma^2(T - t)^3}{3} \right) \]  

(20)

Starting with \( k=0,...,\text{numberOfRealisations}-1 \) samples initialized with value 0, the differential equation models possible evolution (eq.4) of the interest rate. The initialization has been implemented in SamplingSolution.java as

```java
if(Math.abs(kappa)<0.001){
    currentState = new double[numberOfRealisations][1];
    currentState1 = new double[numberOfRealisations][1];
    for (k=0; k<numberOfRealisations; k++)
        currentState[k][0] =0;
   currentState1[k][0]=0;
}
```

where `currentState` is the stochastic and `currentState1` the deterministic component sampling possible evolutions for the interest rate forecast.

Starting from the initial value, possible future realizations of the accumulated interest rate are then calculated using (eq.13) and the expected value of the discount function is readily obtained from (eq.15, eq.18) using an arithmetic average

\[ P(t, T) = \frac{1}{N} \sum_{k=1}^{N} P_k(t, T) \]  

(21)
taking a large number $N$ of possible realizations and replacing the stochastic integral with a random variable $\xi \sim N(0, 1)$ to reproduce the mean and the variance of (eq.17, eq.20). For $a \neq 0$ this yields

$$\frac{\sigma}{a} \int_t^T \left(1 - e^{-a(T-\tau)}\right) d\tau = \xi \sqrt{\frac{\sigma^2}{2a^3} \left(-3 + e^{-2a(T-t)} (-1 + 4e^{a(T-t)} + 2ae^{2a(T-t)}(T-t))\right)}$$

and for $a = 0$

$$\sigma \int_t^T (T - \tau)d\tau = \xi \sqrt{\frac{\sigma^2(T-t)^3}{3}}$$

(22) (23)

The stochastic component has been implemented in MCSSolution.java

```java
if(a==0.){
    for (int k=0; k<numberOfRealisations; k++){
        currentState[k][0] = random.nextGaussian()*Math.sqrt(sigma*sigma*time*time*time/3);
    }else{
    for (int k=0; k<numberOfRealisations; k++){
        currentState[k][0] = random.nextGaussian()*Math.sqrt(sigma*sigma/(2*a*a*a)*((-3+Math.exp(-2*a*time)*(-1+4*Math.exp(a*time)+2*a*Math.exp(2*a*time)*time))));
    }
}
```

while the deterministic component is accounted for in SamplingSolution.java

```java
double timeStep = runData.getTimeStep(); //Run parameters
double a = runData.getRevSpeed();
double sigma = runData.getVolatility();
double b = runData.getRevValue();
if(a==0){
    for (k=0; k<numberOfRealisations; k++){
        //deterministic rates
        currentState1[k][0] = x[j]*time;
        //discount function
        f[j] +=Math.exp(-(currentState1[k][0]+currentState[k][0]));
    }
}else{
    for (k=0; k<numberOfRealisations; k++){
        //deterministic rates
        currentState1[k][0] = x[j]/a*(1-Math.exp(-a*time))+b*time-b/a*(1-Math.exp(-a*time));
        //discount function
    }
}
```
\[
f[j] += \text{Math.exp}(-(\text{currentState1}[k][0]+\text{currentState}[k][0]));
\]
\}
\)
f[j] = f[j]/\text{numberOfRealisations};

For the simple case where the parameters \(a, b, \sigma\) are constants the numerical solution can be compared with an analytical solution for the Vasicek model [1]

\[
P(t, T) = A(t, T)e^{-B(t, T)r_t}
\]

where

\[
B(t, T) = \frac{1 - e^{-a(T-t)}}{a}
\]

and

\[
A(t, T) = \exp\left[\frac{(B(t, T) - T + t)(a^2b - \sigma^2/2) - \sigma^2B(t, T)^2}{4a}\right], \quad a \neq 0
\]

These expressions reduce to \(B(t, T) = T - t\), and \(A(t, T) = \exp[\sigma^2(T - t)^3/6]\) in the limit where \(a \to 0\).

In the java code this is implemented as

\[
\text{if}(a==0)
\]
\[
g[j] = \text{Math.exp}((\text{sigma}\times\text{sigma}\times\text{time}\times\text{time}\times\text{time}/6)+x[j]\times\text{time});
\]
\[
\text{else}
\]
\[
g[j] = \text{Math.exp}(((1 - \text{Math.exp}(-a\times\text{time}))/a-\text{time})\times
\]
\[
(\text{a}\times\text{a}\times\text{b}\times\text{sigma}\times\text{sigma}/2)/(\text{a}\times\text{a})-\text{sigma}\times\text{sigma}^2
\]
\[
((1 - \text{Math.exp}(-a\times\text{time}))/a)^2\times((1 - \text{Math.exp}(-a\times\text{time}))/a)/
\]
\[
(4\times a)^2\times\text{Math.exp}(-(1 - \text{Math.exp}(-a\times\text{time}))/a)\times x[j]);
\]

and will be displayed in a plot with another color (blue).

The VMARKET applet simulates the value of the discount function \(P(r)\) as a function of the current value of the interest rate \(0 < r < 0.2\) for an increasing time to maturity \(T-t<1\). The black dotted line shows the result obtained using the Monte Carlo method and the blue solid line shows the result obtained using the analytic solution.

The parameters are as follows: \text{MeanRevSpeed} is the value of \(a\) measuring the speed of the mean reversion process, \text{MeanRevValue} is the value of \(b\) measuring the mean reversion level, \text{Volatility} is the value of \(\sigma\).

To test the evolution of the discount function first in a very simple case, the VMARKET applet shows what happens without drift / volatility (\text{MeanRevSpeed}=0, \text{MeanRevValue}=0, \text{Volatility}=0). The discount function decreases exponentially in time \(P(t, T) = \exp(-r(T-t))\), as expected for a risk free investment. With one year to maturity (\text{RunTime} = 1) value \(P(t = 0, T = 1) = \exp(-0.1) = 0.905\) is perfectly reproduced also in the applet.
Now test the effect of a large volatility (set artificially large value, e.g. Volatility = 0.2) on the price of a bond with one year to the expiry (RunTime = 1): for the same value of the spot rate $r = 0.1$, the discount function $P(t=0,T=1) = 0.911$, which is larger than the value obtained without volatility. A larger value of the volatility brings about a larger value of the discount function.

Now set the parameters as RevSpeed = 1, MeanRevValue = 0.05, Volatility = 0 to check the effect of the drift. The value of the discount function becomes $P(t=0,T=1) = 0.921$, which is larger since the interest rates are forecasted to drop from the spot rate $r = 0.1$ to a lower value around $b = 0.05$. The effect is further enhanced to $P(t=0,T=1) = 0.930$, when setting MeanRevSpeed=2, which accelerates the process.

### 2.2 Convergence study and validations

Knowing that the Monte-Carlo solution converges as the square-root of the number of walkers, it is easy to calculate the extrapolated value and an error estimate. Choosing the value of parameters $a = 1$ (MeanRevSpeed), $b = 0.05$ (MeanRevValue), $\sigma=0.08$ (Volatility), the discount function is evaluated for a spot rate of $r = 0.1$ using an increasing number of walkers $N$. The results shown in the table below.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\frac{1}{\sqrt{N}}$</th>
<th>$P(t,T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.1</td>
<td>0.928254</td>
</tr>
<tr>
<td>400</td>
<td>0.05</td>
<td>0.920168</td>
</tr>
<tr>
<td>1600</td>
<td>0.025</td>
<td>0.921236</td>
</tr>
<tr>
<td>6400</td>
<td>0.0125</td>
<td>0.921711</td>
</tr>
<tr>
<td>25600</td>
<td>0.00625</td>
<td>0.921947</td>
</tr>
</tbody>
</table>

The points (0.025; 0.921236), (0.0125; 0.921711), (0.00625; 0.921947) lie on the linear function $P = -0.03776 \frac{1}{\sqrt{N}} + 0.922183$. So, when $N \to \infty$, that present value of discount function is $P = 0.922183$. The error estimate is $0.922183-0.921947 = 0.000236$. Thus, $P = 0.922183 \pm 0.000236$. This result agrees with value from the analytic solution $P = 0.922124$, providing the validation sought for the numerical scheme.

### 2.3 Pricing CDS assuming stochastic risk free interest rate

Combine a stochastic risk free interest rate with a constant hazard rate. From an initial value of the CDS equal 0, possible future realizations for the value of the contract are calculated using (eq.3, section 1). In the VMARKET applet, this has been implemented in SamplingSolution.java as
\[ Q = \text{runData.getProbNoDef}(); \]
\[ R = \text{runData.getRecovery}(); \]
\[ \text{RunTime} = \text{runData.getRunTime}(); \]
\[ \text{CouponRate} = \text{runData.getCouponRate}(); \]
\[ \text{CouponFreq} = \text{runData.getCouponFreq}(); \]
\[ \text{double} \ tau = \frac{1}{\text{CouponFreq}}; \]
\[ \text{double} \ n = \text{CouponFreq} \times \text{RunTime}; \]

\[
\text{if}(a==0) \{
\text{for} \ (k=0; \ k<\text{numberOfRealisations}; \ k++)\{
\text{currentState1}[k][0] = x[j] \times \text{time};
\text{f}[j] += \text{Math.exp}(-\text{currentState1}[k][0]+\text{currentState}[k][0]);
\}
\}
\]
\[
\text{else}\{
\text{for} \ (k=0; \ k<\text{numberOfRealisations}; \ k++)\{
\text{currentState1}[k][0] = \frac{x[j]}{a} \times (1-\text{Math.exp}(-a\times\text{time}))+b\times\text{time}-b/a \times (1-\text{Math.exp}(-a\times\text{time}));
\text{f}[j] += \text{Math.exp}(-\text{currentState1}[k][0]+\text{currentState}[k][0]));
\}
\}
\]
\[ \text{f}[j] = \frac{\text{f}[j]}{\text{numberOfRealisations}}; \]
\[
\text{for} \ (k=1; \ k<n+1; \ k++)\{
\text{if} \ (\text{time}=0)
\text{f}[j]=0; \quad \text{//Initial value of CDS is zero}
\text{if} \ (\text{time}<\tau && \text{time} \neq 0)\{
\text{f}[j]=(1-R) \times \text{f}[j] \times (1-Q);
\text{break};
\}
\text{if} \ (\text{time}>k \times \tau && \text{time}<(k+1) \times \tau)\{
\text{f}[j]=(1-R) \times \text{f}[j] \times (1-Q)-k \times \text{CouponRate} \times \text{f}[j] \times Q;
\text{break};
\}
\}
\]

where \( Q \) is the probability of having no default (in the applet it is \text{PrbNoDefault}), \( R \) is the recovery rate (parameter \text{Recovery}), \( \text{CouponRate} \) is annual fixed insurance payment to compensate for the risk of default (\text{CouponRate} in the applet) and \( \text{CouponFreq} \) is the frequency of payment this coupon typically annual (\text{CouponFreq}).

Switch to Value \text{CDS} \text{ with} \text{P(t,T)} \text{ and set the value of parameter to RunTime = 3 years, MeanRevSpeed = 1, MeanRevValue = 0.05, PrbNoDefault = 0.7, Volatility = 0.05, CouponRate = 0.03, CouponFreq = 2, TimeStep = 0.01923077 (one week), Recovery = 0.3. This illustrates the evolution of the CDS value, as a function of the spot rate \( 0 < r < 0.2 \) assuming a mean reversion process to forecast future interest rates \( a = 1, b = 0.1, \sigma = 0.2 \) up to 3 years to maturity, with a semi-annual
payment of 3% insurance coupon, 30% recovery rate and a probability of having no default equal 0.7. The value of CDS is chosen to be zero at the outset of the contract. After one small step, the price of the CDS jumps to account for the initial payment of the insurance coupon and becomes approximately equal to the value, that the buyer of the CDS receives in the credit event (1-recovery rate), multiplied by probability of default during the time interval until the next insurance coupon is payed. As the time increases, the price of the contract decreases, that means, this value is discounted with the stochastic evolution of the interest rate $r$. The price of CDS jumps again after six months ($\Delta t = 0.5$) because of the payments, which buyer has to make at regular time intervals.

Let us experiment to find out how each parameter influences the price of a CDS. For constant interest rates ($\text{MeanRevSpeed} = 0$, $\text{MeanRevValue} = 0$, $\text{Volatility} = 0$) the value of a CDS is $V_{CDS} = 0.062$ for the spot rate $r = 0.1$. Increasing the $\text{Volatility} = 0.1$, the value increases to $V_{CDS} = 0.066$, which is bigger since the discount function evolves with the volatility and changes the value of the CDS. The similar effect can be observed in the presence of mean reverting drifts ($\text{MeanRevSpeed} = 1$, $\text{MeanRevValue} = 0.05$, $\text{Volatility} = 0$): the value increases to $V_{CDS} = 0.069$.

Let us check now, how a constant probability of default changes the value of CDS. Taking $\text{PrbNoDefault} = 0.7$ with a 3 year lifetime of the contract and constant interest rates, the maximum value of the CDS is 0.082. With a higher probability of default, $\text{PrbNoDefault} = 0.5$, the value of the CDS increases to 0.252, which is indeed in agreement with the higher value of the insurance.

Finally, check the effect of the value of the insurance coupon ($\text{CouponRate}$), which is paid on regular time intervals ($\text{CouponFreq}$). Setting $\text{CouponRate} = 0.03$, $\text{PrbNoDefault} = 0.7$ the largest price of CDS is equal to 0.082 with a 3 years lifetime of the contract and constant interest rates. Decreasing the coupon to 0.02, the value of the CDS becomes larger $V_{CDS} = 0.122$, since a smaller premium is now paid for the same insurance protection. The same effect is obtained when decreasing the value of the recovery rate ($\text{Recovery}$), where a lower recovery increases the price of the CDS.
3 The Credit Event Process

3.1 Modelling the default probability

Let us assume that the default process follows a Poisson distribution with stochastic intensities or hazard rate \( h(t) \). In the literature this is called a Cox process (see ref. [4] for more details). The hazard rate is defined so that \( h \Delta t \) is the probability of default between times \( t \) and \( t + \Delta t \) as seen at time \( t \) and conditional on no earlier defaults. Moreover, define \( Q(h, t) \Delta t \) as the probability that no default occurs between time \( t \) and \( t + \Delta t \) as seen at time zero:

\[
Q(h, t) = \exp \left( - \int_0^t h \, d\tau \right)
\]  
(27)

where \( h_\tau \) is the instantaneous forward rate of default at time \( \tau \).

Finally, assume that the default intensity process obeys the stochastic equation

\[
d \ln(h_t) = a(b - \ln(h_t))dt + \sigma dz
\]  
(28)

where \( h_t \) is the default intensity, \( a(b - \ln(h_t)) \) is a mean reversion, \( \sigma \) is the variance or volatility in the hazard rate, \( dz \) is a Wiener process, and \( a, b, \sigma \) are constants used to forecast the intensity of default with a log-normal distribution.

The evolution of the intensity of the default in a volatile market can be simulated using the Monte-Carlo method. Let us convert the continuous-time model (eq.28) to discrete-time. Applying rules from conventional differential calculus to \( e^{at} \ln(h_t) \) and substituting (eq.28), give:

\[
d e^{at} \ln(h_t) = a e^{at} \ln(h_t)dt + e^{at} \frac{1}{h_t} dh_t
\]  
(29)

\[
= a e^{at} \ln(h_t)dt + e^{at} [a(b - \ln(h_t))dt + \sigma dz_t]
\]  
(30)

\[
= ab e^{at} dt + \sigma e^{at} dz_t
\]  
(31)

This implies that

\[
e^{at} \ln(h_t) - e^{a(t-\Delta t)} \ln(h_{t-\Delta t}) = ab \int_{t-\Delta t}^{t} e^{\sigma \tau} d\tau + \sigma \int_{t-\Delta t}^{t} e^{\sigma \tau} dz_\tau
\]  
(32)

\[
= b \left( e^{at} - e^{a(t-\Delta t)} \right) + \sigma \int_{t-\Delta t}^{t} e^{\sigma \tau} dz_\tau
\]  
(33)

where the last term

\[
\sigma \int_{t-\Delta t}^{t} e^{\sigma \tau} dz_\tau \text{ has a normal distribution } N \left( 0, \frac{\sigma^2}{2a} e^{2at} (1 - e^{-2a\Delta t}) \right), a \neq 0.
\]  
(34)

Identify the differential and exponentiate to obtain the default intensity

\[
h_t = \exp \left( \ln(h_{t-\Delta t})e^{-a(\Delta t)} + b \left( 1 - e^{-a\Delta t} \right) + \xi \sqrt{\frac{\sigma^2}{2a} (1 - e^{-2a\Delta t})} \right) \text{ for } a \neq 0
\]  
(35)
and
\[ h_t = h_{t-\Delta t} \exp \left( \xi \sqrt{\sigma^2 \Delta t} \right) \quad \text{for } a = 0 \] (36)

here \( \xi \sim N(0,1) \) is a normally distributed random number generated anew for each time step \( \Delta t \). In a Monte-Carlo calculation, start with \((k=0, \ldots, \text{numberOfRealisations}-1)\) samples modelling possible evolutions of the intensity \( (\text{currentState}[k][j]) \) initialized in the applet with the initial value \( x[j] \)

```
currentState = new double[numberOfRealisations][mesh.size()];
currentState1 = new double[numberOfRealisations][mesh.size()];
for (j=0; j<mesh.size(); j++)
    for (k=0; k<numberOfRealisations; k++){
        currentState[k][j] = x[j];
        currentState1[k][j]=x[j];
    }
```

Possible future realizations of the intensity are calculated in MCSSolution.java according to (eq.35, eq.36) as

```
double timeStep = runData.getTimeStep(); //Run parameters
double a = runData.getRevSpeed();
double sigma = runData.getVolatility();
double b = runData.getRevValue();
for(int j=0; j<f.length; j++) {
    if (a==0){
        for (int k=0; k<numberOfRealisations; k++){
            currentState[k][j]=currentState[k][j]*Math.exp(
                random.nextGaussian() * Math.sqrt(sigma*sigma*timeStep));
            currentState1[k][j] += currentState[k][j];
        }
    }else{
        for (int k=0; k<numberOfRealisations; k++){
            //Deterministic term
            currentState[k][j]=Math.exp(Math.log(currentState[k][j])*Math.exp(-a*timeStep)+b*(1-Math.exp(-a*timeStep)))*
            //Stochastic term
            Math.exp(random.nextGaussian() * Math.sqrt(sigma*sigma/(2*a)*(1-Math.exp(-2*a*timeStep))));
            currentState1[k][j] += currentState[k][j];
        }
    }
}
```
It is easy to identify the deterministic and the random component of the evolution in (eq.35). The variable `currentState1[k][j]` accumulates possible realizations of the value of the intensity to prepare for the integral involved in (eq.27). The Monte-Carlo method uses a large number of possible realizations as an maximum likelihood estimator for the mean value of the probability

\[ Q(h, t) = \frac{1}{N} \sum_{k=1}^{N} Q_k(h, t) \]  

(37)

The value of the probability that no default occurs during the time interval [0,t] is finally calculated using a trapezoidal rule

\[ \int_{a}^{b} h(t)dt \approx q \left[ \sum_{j=1}^{n} h(t_j) - \frac{h(a) + h(b)}{2} \right] \]  

(38)

where

\[ q = \frac{b - a}{n} \quad t_j = a + \frac{j}{n} (b - a) \]  

(39)

Hence, using (eq.27)

\[ Q(h, t) = \exp \left[ -\Delta t \left( \sum_{t_j=0}^{N} h(t_j) - \frac{h(0) + h(t)}{2} \right) \right] \]  

(40)

The scheme has been implemented in SamplingSolution.java using the random walk that has previously been computed according to (eq.35, eq.36) to obtain the terminal value of the underlying \( h_j \) in `currentState[k][j]`. The average (eq.37) is then performed by summation over all the possible realizations:

```java
for (k=0; k<numberofRealisations; k++){
    //accumulate probability for each walker
    //trapezoidal rule
    f[j] += Math.exp(-timeStep*currentState1[k][j]+timeStep*x[j]/2+timeStep*currentState[k][j]/2);
}
f[j] = f[j]/numberofRealisations; //average
```

The following parameters control the calculation in the VMARKET applet: MeanRev Speed is the value of \( a \), MeanRevValue is the value of \( b \), Volatility is the value of \( \sigma \). The applet illustrates the value of the probability (the black dotted line) as a function of the default intensity for an increasing duration of the contract \( t \) (RunTime). The blue solid line shows the result obtained analytically for a probability \( Q(t) = \exp(-ht) \), when \( h \) is constant and is in good agreement with the value
predicted in the model for a constant intensity ($MeanRevSpeed = 0$, $MeanRevValue = 0$, $Volatility = 0$): the black line covers the blue line as expected.

To visualize the dynamics implied by a stochastic evolution of the hazard rate, take large values for the drift $MeanRevSpeed = 1$, $MeanRevValue = 0.35$ and the $Volatility = 0.2$. Observe how the probability of no default decreases with time below the value calculated analytically (the black line drops below the blue line in the figure 6).

To investigate the effect of a stochastic hazard rate alone, set $MeanRevSpeed = 0$, $MeanRevValue = 0$ and artificially large value of $Volatility = 1$. The probability with stochastic intensities (the black line) drops below the value obtained for a constant hazard rate (the blue line), showing as expected that the default probability increases with the volatility in the hazard rate.

### 3.2 Pricing CDS for a stochastic hazard rate

Assume a stochastic default intensity (hazard rate) and a constant risk free interest rate. Choosing the initial value of the CDS $V_{CDS}(t = 0) = 0$ possible future realizations can be calculated from (eq.3, section 1) using the Monte-Carlo model for $Q(h,t)$ developed in the previous section. This pricing has been implemented in SamplingSolution.java as

```java
r = runData.getSpotRate();
R = runData.getRecovery();
RunTime = runData.getRunTime();
CouponRate = runData.getCouponRate();
CouponFreq = runData.getCouponFreq();
double tau = 1/CouponFreq;
double n = CouponFreq*RunTime;

for (k=0; k<numberOfRealisations; k++){
    f[j] += Math.exp(-timeStep*currentState1[k][j]+timeStep*x[j]/2+timeStep*currentState[k][j]/2);
}

for (k=0; k<n+1; k++){
    if (time==0)
        f[k]=0;
    if (time<tau && time != 0){
        f[k]=(1-R)*Math.exp(-r*time)*(1-f[k]);
        break;
    }
    if (time>i*tau && time<(k+1)*tau){
        f[k]=(1-R)*Math.exp(-r*time)*(1-f[k])-
```
In the code, \( r = \text{SpotRate} \) is the spot rate, \( R = \text{Recovery} \) is the recovery rate, \( \text{CouponRate} \) is annual fixed insurance payment to compensate for the risk of default and \( \text{CouponFreq} \) is the frequency of payment this coupon.

To illustrate the properties of this model, assume a CDS with semi-annual 7\% coupon, a life time of 3 years, a 50\% recovery rate and a fixed 5\% interest rate. In the applet select Value CDS with \( Q(t) \) and set the value of parameter as \( \text{RunTime} = 3, \text{MeanRevSpeed} = 0.1, \text{MeanRevValue} = 0.1, \text{Volatility} = 0.1, \text{SpotRate} = 0.05, \text{CouponRate} = 0.07, \text{CouponFreq} = 2, \text{TimeStep} = 0.01923077, \text{Recovery} = 0.5. \) The value of the contract is also displayed in figure 8 as a function of the initial default intensity \( (a = 0.1, b = 0.1, \sigma = 0.1) \) with 3 years to the expiry date. The accumulated probability of default is small at the beginning but grows with time making the CDS more valuable. The value jumps down when the buyer pays the fixed periodical amount for his insurance.

To further develop an intuition for this model take the present (spot) hazard rate of \( h = 0.202 \) displayed on the horizontal axis. In the simplest case where the hazard rate remains constant, \( (\text{MeanRevSpeed} = 0, \text{MeanRevValue} = 0 \text{ and } \text{Volatility} = 0) \), the value of the contract becomes positive for \( h > 0.202 \) and negative for \( h < 0.202 \) showing that the CDS can be an asset (in a risky market) or a liability (when there is little risk). For the case of a large \( \text{Volatility} = 0.4 \), figure 9 shows that the neutral point of the contract (where the \( V_{CDS}(h^*) = 0 \)) decreases to \( h^* = 0.190 \). This is consistent with the previous observation that the default probability increases with the volatility in the hazard rate making the insurance valuable. Let us examine the result in the presence of mean reverting drifts \( (\text{MeanRevSpeed} = 0.1, \text{MeanRevValue} = 0.35, \text{Volatility} = 0) \): the neutral point is moved to \( h^{**} = 0.150 \), since the larger value of the CDS results from the higher probability of default.

To check the effect of the interest rate on the value of a CDS with constant intensities, vary the \text{SpotRate} from 0.05 to 0.2: the maximum value of contract decreases from \( V_{CDS_{max}} = 0.334 \) to \( V_{CDS_{max}} = 0.213 \). The contract drops in value with the discounting at a higher interest rate.

To finally test the effect of fixed coupons \( (\text{CouponRate} = 0.07) \) paided twice a year \( (\text{CouponFreq} = 2) \), take a constant default intensity \( (\text{MeanRevSpeed} = 0, \text{MeanRevValue} = 0, \text{Volatility} = 0) \) and a constant interest rate \( (\text{SpotRate} = 0.05) \): the neutral point of the contract is \( h = 0.202 \). Reducing the \text{CouponRate} to 0.05 moves the neutral point to lower value \( h^{***} = 0.160 \) making the insurance provided by the CDS more valuable. Similar conclusions can be drawn when the recovery rate is reduced.
4 Pricing CDS using a two-factor model

Combine a stochastic interest rate with a stochastic hazard rate, using a normal process for the interest rate and a log-normal process for the hazard rate:

\[ dr = a_r(b_r - r)dt + \sigma_r dz_1 \]  
\[ d \ln(h) = a_h(b_h - \ln(h))dt + \sigma_h dz_2 \]

This two-factor model assumes no correlation between these two components. The value of the discount function can therefore be calculated exactly as in section 2: for \( a_r \neq 0 \)

\[ P(t, T) = E \left[ \exp \left( -r_t \frac{1}{a_r} \left( 1 - e^{-a_r(T-t)} \right) - b_r(T - t) + \frac{b_r}{a_r} \left( 1 - e^{-a_r(T-t)} \right) - \frac{\sigma_r}{a_r} \int_t^T \left( 1 - e^{-a_r(T-\tau)} \right) d\tau \right) \right] \]

where the term

\[ \frac{\sigma_r}{a_r} \int_t^T \left( 1 - e^{-a_r(T-\tau)} \right) d\tau \]

follows a normal distribution

\[ N \left( 0, \frac{\sigma_r^2}{2a_r^3} \left( -3 + e^{-2a_r(T-t)} \left( -1 + 4e^{a_r(T-t)} + 2ae^{2a_r(T-t)(T - t)} \right) \right) \right) \]

In the limit when \( a_r \to 0 \), these expressions reduce to

\[ P(t, T) = E \left[ \exp \left( -r_t(T - t) - \sigma_r \int_t^T (T - \tau) d\tau \right) \right] \]

where

\[ \sigma_r \int_t^T (T - \tau) d\tau \]

has a normal distribution

\[ N \left( 0, \frac{\sigma_r^2(T - t)^3}{3} \right) \]

The probability of no default is obtained as in previous section.

\[ Q(h, t) = \exp \left[ -\Delta t \left( \sum_{t_j=0}^{N} h(t_j) - \frac{h(0) + h(t)}{2} \right) \right] \]

where

\[ h_t = \exp \left( \ln(h_{t-\Delta})e^{-ah_t(\Delta t)} + b_h \left( 1 - e^{-ah_t(\Delta t)} \right) + \xi \sqrt{\frac{\sigma_h^2}{2ah}} \left( 1 - e^{-2ah\Delta t} \right) \right) \quad \text{for} \ a_h \neq 0 \]
\[ h_t = h_{t-\Delta} \exp \left( \xi \sqrt{\sigma_h^2 \Delta t} \right) \quad \text{for } a_h = 0 \quad (51) \]

where \( \xi \sim N(0,1) \).

Start with \((k=0,\ldots,\text{numberOfRealisations}-1)\) samples to forecast possible evolutions of the interest rate \((\text{currentState1}[k][j])\) and the intensity \((\text{currentState2}[k][j])\): the first is initialized with the values \(x[j]\) corresponding to the horizontal axis of the applet and the second is initialized with a constant value of either the spot rate or the default intensity depending on the type of the 2D output plot in the applet. This initialization has been implemented in SamplingSolution.java as

\[
\text{currentState} = \text{new double[numberOfRealisations][mesh.size()]};
\text{currentState1} = \text{new double[numberOfRealisations][mesh.size()]};
\text{currentState2} = \text{new double[numberOfRealisations][mesh.size()]};
\text{currentState3} = \text{new double[numberOfRealisations][mesh.size()]};
\text{for} \ (j=0; \ j<\text{mesh.size();} \ j++)
\quad \text{for} \ (k=0; \ k<\text{numberOfRealisations;} \ k++)\{
\quad \quad \text{currentState}[k][j] = x[j];
\quad \quad \text{currentState1}[k][j] = x[j]; // Interest rate
\quad \quad \text{currentState2}[k][j] = x[j]; // Default intensity
\quad \quad \text{currentState3}[k][j] = x[j];
\}
\]

Possible future realizations for the values of the intensity and the interest rate are then calculated in MCSSolution.java as

\[
\text{for(int j=0; j<f.length; j++) } \;
\quad \text{for (int k=0; k<\text{numberOfRealisations}; k++){}
\]

\[
\quad \text{if (a==0)}\{
\]

\[
\quad \quad /* \text{interest rate} */
\quad \quad \text{currentState1}[k][j]=\text{currentState}[k][j]*time+
\quad \quad \quad \text{random.nextGaussian()}*\text{Math.sqrt}((\text{sigma}\text{*sigma})*time*\text{time}/3);
\]

\[
\quad \quad /* \text{intensity of default} */
\quad \quad \text{currentState2}[k][j]=\text{currentState2}[k][j]*\text{Math.exp}(
\quad \quad \quad \text{random.nextGaussian()}*\text{Math.sqrt}((\text{sigma}\text{2}\text{*sigma}\text{2}*\text{TimeStep}));
\quad \quad \text{currentState3}[k][j] += \text{currentState2}[k][j];
\]

\[
\quad \} \text{else}\{
\]

\[
\quad \quad /* \text{interest rate} */
\quad \quad \}
\]

15
currentState1[k][j] = currentState[k][j] / a * (1 - Math.exp(-a * time)) + b * time - b / a * (1 - Math.exp(-a * time)) +
    random.nextGaussian() * Math.sqrt(sigma * sigma / (2 * a * a * a) *
    ((-3 + Math.exp(-2 * a * time) * (-1 + 4 * Math.exp(a * time) +
    2 * a * Math.exp(2 * a * time)))));

// intensity of default */
currentState2[k][j] = Math.exp(Math.log(currentState2[k][j]) *
    Math.exp(-a2 * timeStep) + b2 * (1 - Math.exp(-a2 * timeStep))) *
    Math.exp(random.nextGaussian() * Math.sqrt(sigma2 * sigma2 /
    (2 * a2) * (1 - Math.exp(-2 * a2 * timeStep))));
currentState3[k][j] += currentState2[k][j];
}
}
}

The value of the CDS can now be calculated from (eq. 3, section 1) using the dis-
count function (eq. 43, eq. 46) and the probability of no default (eq. 49). It has been
implemented in SamplingSolution.java as

for (k = 0; k < numberOfRealisations; k++){

    // Discount function
    fm[j] += Math.exp(-(currentState1[k][j]));

    // Probability of no default
    timeStep * currentState2[k][j] / (2.));
}

fm[j] = fm[j] / numberOfRealisations;
fp[j] = fp[j] / numberOfRealisations;

// Value of CDS
for (k = 1; k < n + 1; k++){
    if (time == 0)
        f[j] = 0;
    if (time < tau && time != 0){
        f[j] = (1 - R) * fm[j] * (1 - fp[j]);
        break;
    }
    if (time > k * tau && time < (k + 1) * tau){
        f[j] = (1 - R) * fm[j] * (1 - fp[j]) -
            k * CouponRate * fm[j] * fp[j];
        break;
}
where $R$ is the recovery rate (applet parameter `Recovery`), `CouponRate` is annual fixed insurance payment to compensate for the risk of default and `CouponFreq` is the frequency of payment this coupon.

For validation purposes calculate the value of a CDS with constant interest rate ($a_r = 0, b_r = 0, \sigma_r = 0$) and constant default intensity ($a_h = 0, b_h = 0, \sigma_h = 0$) up to ($\text{RunTime} = 3$) years to the expiry. The applet displays the result from 2D model where the spot rate is fixed ($\text{SpotRate} = 0.05$) and the value of the CDS is plotted as a function of the default intensity ($0 < h < 0.7$). The results obtained in this limiting case are in good agreement with the simple result obtained in section 3 (1D model), where the neutral point of the contract (where $V_{\text{CDS}}(h) = 0$) is around $h = 0.202$ and the maximum $V_{\text{CDSmax}} = 0.334$. Increasing the volatility in the hazard rate to a large value, i.e. $\text{Volatility} = 0.4$, the neutral point of the CDS is $h^* = 0.190$ again reproducing what was obtained in the previous section.

Let us examine the parametric dependences in the presence of mean reverting drifts for both the interest rate ($a_r = 1, b_r = 0.05$) and the hazard rate ($a_h = 0.1, b_h = 0.35$) using finite volatilities ($\sigma_r = 0.05, \sigma_h = 0.2$). The result displayed in figure 10 shows that the neutral point of the CDS is located around $h^{**} = 0.129$ with $V_{\text{CDSmax}} = 0.354, V_{\text{CDSmin}} = -0.312$. The same calculation with a 1D model assuming a constant 5% interest rate gives the same neutral point (discounted value of zero is equal to zero for every value of the interest rate) but a lower maximum value $V_{\text{CDSmax}} = 0.349$ and larger minimum value $V_{\text{CDSmin}} = -0.256$, since in 2D model the value of the interest rate increases during the lifetime of the contract.

Figure 11 shows the value of the CDS plotted as a function of the spot rate ($0 < r < 0.2$) and a present value of the hazard rate $h = 0.3$, assuming mean reverting forecast models for both the interest rate ($a_r = 1, b_r = 0.05, \sigma_r = 0.05$) and the hazard rate ($a_h = 0.1, b_h = 0.35, \sigma_h = 0.2$). The results obtained in this case show that $V_{\text{CDSmax}} = 0.112$ and $V_{\text{CDSmin}} = 0.060$. Assuming a probability of no default equal to 0.41, which corresponds to 0.3 of the default intensity, a 1D model of the interest rate predicts that $V_{\text{CDSmax}} = 0.110$ and $V_{\text{CDSmin}} = 0.090$, which are clearly different here, due to the evolution of the hazard rate during the lifetime of the CDS.
5 Conclusion

This report examined two factors that are important for the valuation of a credit default swap (CDS): the stochastic evolution of both the interest rate and the default intensity. Keeping one of the factors fixed, 1D models have been validated against simple limiting cases. For the case of a large volatility in the interest rates of 10%, studies showed that the value of a typical CDS can increase up to 6% when it is compared with fixed interest rates. In the same manner a standard deviation of the hazard rate of 40% (typical of a speculative CCC bond) decreases the neutral point (where the contract is worthless) by 6%; this shows that a stochastic component in the hazard rate decreases the expected default and increases the value of the insurance contract. Because of the compounding of interest and the hazard rates, mean reverting drifts can, under the circumstances, be even more important.

Combining stochastic hazard and interest rates into a 2D model yields a value that is considerably different from the single factor models. For the same parameters as in the previous section, the value obtained from the 1D interest rate model can be more than 60% lower than the CDS value calculated with the 2D model. The largest difference is obtained for the one factor model of interest rate where the default probability does not evolve during the lifetime of the contract.

Note that this report neglected the correlation between the interest rate and the hazard rate. This is an obvious limitation that could be removed by making the default intensity \( h = h(\sigma, r) \) dependent on the interest rate in (eq.42 section 4). This complicates the evaluation of the integral (eq.32 section 3) and could not be finished during the limited time that was allocated for this project. The subject should however be of sufficient interest to warrant more detailed studies.
Fixed leg

Premium payments until default occurs

Par value minus recovery rate

Default leg

Figure 1: The credit default swap (CDS) cash flow structure

Figure 2: Vasicek model for the value of the discount function $P(r)$ as a function of the spot rate $0 < r < 0.2$ one year before the maturity date $T - t = 1$. Case with large value of mean reversion speed (MeanRevSpeed=2, MeanRevValue=0.05).
Figure 3: Same as figure 2. Case with large value of the drift and the volatility (MeanRevSpeed=1, MeanRevValue=0.05, Volatility=0.08).

Figure 4: The value of the credit default swap, as a function of the spot rate using a mean revers model to forecast the interest rate ($a = 1$, $b = 0.05$, $\sigma = 0.05$). The value is given 3 years before the expiry date, assuming a semi-annual payment of 3% insurance coupon, 30% recovery rate and a probability of having no default equal to 0.7.
Figure 5: Same as figure 4. Case where the interest rate is forecasted to remain constant ($a = 0$, $b = 0$, $\sigma = 0$) and a probability of having no default equal to 0.5.

Figure 6: The value of the probability (the black dotted line) as a function of the default intensity with large value of the drift and the volatility (MeanRevSpeed = 1, MeanRevValue = 0.35, Volatility = 0.2). The blue solid line corresponds to results obtained analytically when $h$ remains constant.
Figure 7: Same as figure 6. Case without drift and with large value of the volatility (MeanRevSpeed = 0, MeanRevValue = 0, Volatility = 1).

Figure 8: The value of the CDS as a function of the present value of the default intensity (0 < h < 0.7) assuming a mean reversion model to forecast future intensities (a = 0.1, b = 0.1, σ = 0.1), a semi-annual 7% coupon, a 3 years life time, 50% recovery rate and a fixed 5% interest rate. The value of the CDS (black dotted line) intersects the horizontal axis at the neutral point $h^* = 0.157$. 
Figure 9: Same as figure 8. Case of a large variance in the hazard rate \(a = 0, b = 0, \sigma = 0.4\). The value of the CDS intersects the horizontal axis at the neutral point \(h^* = 0.190\).

Figure 10: The value of the CDS obtained from a 2D model when one dimension is fixed (the spot rate is equal to 5%) and the CDS value is plotted as a function of the default intensity \((0 < h < 0.7)\). The forecasted default intensity and the interest rate evolve with the drift and the volatility \((a_h = 0.1, b_h = 0.35, \sigma_h = 0.2, a_r = 1, b_r = 0.05, \sigma_r = 0.05)\) and with no correlation between the components.
Figure 11: The value of the CDS obtained from a 2D model when one dimension is fixed (the default intensity is equal to 0.3) and the CDS value is plotted as a function of the interest rate ($0 < r < 0.2$). The forecasted default intensity and the interest rate evolve with the drift and the volatility ($a_h = 0.1$, $b_h = 0.35$, $\sigma_h = 0.2$, $a_r = 1$, $b_r = 0.05$, $\sigma_r = 0.05$) and with no correlation between the components.
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