Pricing Convertible Bonds using Stochastic Interest Rate

A comparison using Hull-White, Ho-Lee and a deterministic model

by

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Thomas Nordqvist

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Abstract

The purpose of this thesis is to compare the pricing of convertible bonds when using different short rate models.

A convertible bond is a bond that may be converted into stocks. Just like a normal bond it has a face value that will be paid back at the date of maturity. The holder has the right to convert the bond into a predetermined number of shares in the company in a given period of time.

For a European convertible bond, there is no possibility for early conversion, there exists an analytical expression. Since contract conditions on the market varies a lot a numerical method using Crank-Nicholson scheme has been used in this thesis.

A stochastic treatment of the short interest rate is appropriate when dealing with financial contracts with long maturity such as convertible bonds.

In this thesis two stochastic models are used to model the short interest rate, the Hull-White and the Ho-Lee model. The results using stochastic short rate are compared to a model using a deterministic short rate. The effect of the correlation between the stock price and the short interest rate is taken into consideration. The markets opinion on the specific company is used to model the credit risk.

A comparison between deterministic and stochastic models shows that for a long time to maturity the choice if interest rate model is significant. As expected, the difference between the models is not so relevant for shorter time to maturity.
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Chapter 1

Introduction

A convertible bond is a bond that may be converted into stocks. Just like a normal bond it has a face value that will be paid back at the date of maturity. The holder has the right to convert the bond into a predetermined number of shares in the company in a given period of time.

The convertible bond may be considered as a combination of a bond and a European call option on the stock when there is no reason for early conversion. A commonly used pricing method for convertible bonds may be to price the bond and stock option as two separate contracts. However this is not an appropriate method for all convertible bond contracts. For example if the last day of conversion is not the same as the maturity date the choice to convert or not depends on the stock price and the interest rate. Another example is if there is a dividend on the stock then early conversion may be of interest.

Due to the various contract conditions in a convertible bond a corresponding pricing method should be able to handle stock dividends, coupons and early conversion.

When pricing convertible bonds the interest rate is a variable of great importance. Traditionally, the interest rate has been considered to be deterministic but time dependent. Using a deterministic interest rate can be an acceptable approximation for contracts with a short lifetime. However, convertible bonds often have a time to maturity up to 5 years. Therefore, the volatility of the interest rate could have an impact of the price of the contract.

The objective of this master thesis is to price convertible bonds using different stochastic interest rate processes. The different models will be compared with a deterministic model to determine the impact of the choice of the model on the final price.
Since there exists no analytical formula for American convertible bonds a numerical method is needed. In this thesis the Crank-Nicholson method for 3 dimensions is used which is a finite difference method.

In this thesis both stock price and short rate will be considered as stochastic processes. The stock price process will be modelled as a geometric Brownian motion. Market data for stock options will be used to estimate the stock price volatility. Two different stochastic processes will be used to model the short rate, the Ho-Lee and the Hull-White model. Both models follow the future expected short rate curve adding a stochastic term. The Hull-White also includes a mean reverting feature, which pulls the interest rate back to its expected value. Nelson-Siegel’s model together with market bond data will be used to calculate the short rate curve. To decide the parameters in the Ho-Lee and Hull-White process bond option prices and historical bond prices will be used. The correlation between the stock price process and the short rate process and its effect on the convertible bond price will be examined.

Since a convertible bond is issued by a company the credit risk of that company has an effect on the convertible bond price. By creating a short rate curve for each issuing company the credit risk will be taken into account. Each company will then have its own short rate process.

All the work with this thesis has been made jointly by the two authors Thomas Nordqvist and Martin Öhrn. The chapters 2 and 4 are mainly written by Thomas Nordqvist and chapters 3 and 5 are mainly written by Martin Öhrn. Chapters 1 and 6 are jointly written.
Chapter 2

Financial assets background

To be able to price convertible bonds and other contracts with stochastic stock price and interest rate a background in stochastic calculus is needed. A knowledge about the stock price process, options, bonds and interest rate processes will be necessary before moving on to convertible bonds.

2.1 Stochastic process

A stochastic process is a parametrized collection of random variables

\[ \{X_t\}_{t \in T} \]

defined on a probability space \((\Omega, \mathcal{F}, P)\) and assuming values in \(\mathbb{R}\).

For each \(t \in T\) fixed there is a random variable \(\omega \rightarrow X_t(\omega); \omega \in \Omega\), when fixing \(\omega\) the function \(t \rightarrow X_t(\omega)\) is called a path of \(X_t\). One can think of \(t\) as time and each \(\omega\) as an individual experiment.

A Wiener process is a stochastic process and defined by

1. \(W(0) = 0\)
2. \(W(t)\) has independent increments
3. \(W(t) - W(s)\) is Gaussian \(N(0, t-s)\) for \(s < t\)
4. \(W(t)\) has continuous trajectories.

To introduce uncertainty in differential equations, a common model is to consider

\[
\begin{align*}
\frac{dX(t)}{dt} &= \alpha(X(t), t)dt + \beta(X(t), t)dW(t) \\
X(0) &= X_0
\end{align*}
\]

meaning that \(X\), the stochastic process solution of the above satisfies

\[ X(t) = X_0 + \int_0^t \alpha(X(s), s)ds + \int_0^t \beta(X(s), s)dW(s), \forall t \in [0, T]. \]
In financial mathematics Itô’s formula is applied on stochastic processes, for example on a stock price process. A complete derivation of Itô’s formula can be seen in Øksendal [7].

Assume that \( X \) satisfies equation 2.1 and let \( f \) be a given function. Then applying Itô’s formula on \( y(t) \equiv f(X(t), t) \) yields

\[
dy(t) = \left( \frac{\partial f(X(t), t)}{\partial t} + a(X(t), t) \frac{\partial f(X(t), t)}{\partial X} + \frac{\beta^2(X(t), t)}{2} \frac{\partial^2 f(X(t), t)}{\partial X^2} \right) dt \\
+ \beta(X(t), t) \frac{\partial f(X(t), t)}{\partial X} dW(t).
\]

The Itô formula can be naturally generalized to the case where the function \( f \) depends on more than one stochastic process e.g. \( f(X, Y, t) \).

### 2.2 Stock price process

The most widely used model to represent the stock price process is to let the stock evolve as a geometric Brownian motion

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW(t), \\
S(0) = S_0,
\]

where \( \mu \) is the drift, and \( \sigma \) the volatility of the stock.

To determine the stock price process equation 2.3 has to be solved. Divide the equation with \( S(t) \).

\[
dS(t)/S(t) = \mu dt + \sigma dW(t).
\]

Using Itô’s formula as in section 2.1 on the expression above, substitute the \( dS(t) \) term and integrate gives that

\[
d\ln(S(t)) = dS(t)/S(t) - \frac{1}{2} \frac{\sigma^2 S(t)}{S^2(t)} dt \\
= \mu dt - \frac{\sigma^2}{2} dt + \sigma dW(t) \\
\ln\left( \frac{S(t)}{S(0)} \right) = \mu t - \frac{\sigma^2}{2} t + \sigma W(t) \\
S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2}) t + \sigma W(t)}.
\]

\( S(t) \) has a lognormal distribution so the expected value of \( S(t) \) is \( E[S(t)] = S_0 e^{\mu t} \)
and the variance is \( V[S(t)] = S_0^2 e^{2\mu t} \left[ e^{\sigma^2 t} - 1 \right] \).
2.3 Black-Scholes equation

Arbitrage pricing theory
The derivation of the Black – Scholes equation will show that a derivative can be replicated by constructing a portfolio of a combination of the underlying asset and the risk-free asset. This portfolio is said to replicate the derivative security i.e. it behaves exactly as the derivative. The proportions of stock and risk-free asset in the portfolio must be adjusted continuously with time, but no additional money can be added or taken away. The portfolio is said to be self-financing. This replication can be carried out in order to construct a synthetic derivative using underlying and the risk-free assets.

A term often used in the financial market is the concept of arbitrage. Arbitrage can be defined as

\[ V(0) = 0, \]
\[ V(T) > 0, \text{P.a.s.} \]

i.e. \( P(V(T) > 0) = 1 \) where \( V \) is a self-financing portfolio. In pricing theory one has to make the assumption that the market is arbitrage-free, there are no arbitrage possibilities. Assume also the existence of a risk-free paper \( B \) which follows \( dB = rBdt \) where \( r \) is the risk-free interest rate.

Consider a portfolio \( \Pi \) consisting of a sold contract \( f(S,t) \) plus \( \Delta \) quantity of the asset \( S \) plus \( \Lambda \) quantity of the risk-free paper \( B \) i.e. \( \Pi = -f + \Delta S + \Lambda B \). The asset \( S \) is modelled as a geometric Brownian motion, see section 2.2 i.e. \( dS(t) = \mu S(t)dt + \sigma S(t)dW(t) \). Differentiating and the use of Itô’s formula on \( \Pi \) gives

\[
d\Pi = -df + \Delta dS + \Lambda dB \\
= -\left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt - \sigma S \frac{\partial f}{\partial S} dW + \\
\quad + \Delta \left( \mu S dt + \sigma S dW \right) + \Lambda rBdt \\
= -\left( \frac{\partial f}{\partial t} + \mu S \left( \frac{\partial f}{\partial S} - \Delta \right) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - \Lambda rB \right) dt - \sigma S \left( \frac{\partial f}{\partial S} - \Delta \right) dW
\]

By choosing \( \Delta = \frac{\partial f}{\partial S} \) the randomness in the expression \( d\Pi \) is eliminated. This results in a portfolio whose increment is fully deterministic i.e.

\[
d\Pi = -\left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - \Lambda rB \right) dt.
\]

In a market of no-arbitrage this increment should be the same as investing the portfolio value and earning the risk free rate \( d\Pi = r\Pi dt \) i.e.

\[ \Pi dt = r\Pi dt \Leftrightarrow -\left( \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - \Lambda rB \right) dt = r \left( -f + \Delta S + \Lambda rB \right) dt. \]
By looking at time interval $dt$ the following equation has to be solved,

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} - rf = 0.$$ (2.5)

This is the **Black-Scholes partial differential equation**. It has many solutions corresponding to all different contracts that could be defined with $S(t)$ as a underlying process. It should also include initial and/or boundary conditions of the financial contract e.g. $f = \max[S - X, 0]$ when $t = T$ to make the solution unique.

Equation 2.5 is of the form that can be solved by using Feyman – Kač, see Björk [4], and the solution is given by

$$\Psi(t, s) = e^{-r(T-t)} E_{t,s}^Q(\Phi(S(T))).$$

where the $S$ process is defined as in section 2.2 but with the drift $r$ instead of $\mu$ i.e. $dS = rSdt + \sigma S d\hat{W}$, at time $t$ with stock price $s$.

Observe that the expectation is under a probability measure called $Q$-measure. This $Q$-measure is also called the **risk neutral measure**. The normal probability $P$-measure is called the **objective probability measure**. The economic interpretation of the $Q$-measure is that the expected value of any contract is discounted with the risk free rate.

### 2.4 Option theory

An option is a contract on an underlying asset such as a stock, rate, commodity or a currency. An option has some specific contract conditions.

1. There are two basic types of options.

   A **call option** gives the owner of the option the right to buy the underlying asset at a given price.

   A **put option** gives the owner the right to sell the underlying asset at a given price.

2. An option has a maturity date $T$ which describes the lifetime of the contract.

3. There is a feature of the contract that describes when it can be exercised. An American contract can be exercised at any time. A European contract can be exercised at the maturity date only. There are also Bermudan contracts that can be exercised on a finite number of occasions.

4. The exercise price $X$, also called strike, specifies at what price the underlying can be bought or sold.
The price for a European option on the date of maturity $T$ can be described as the payoff

$$P_c(T) = \max [S_T - X, 0],$$
$$P_p(T) = \max [X - S_T, 0].$$

An important relationship between call- and put options is known as put–call parity. Having a portfolio consisting of one European call option plus the present value of the amount $X$ to be received on the maturity date and another portfolio consisting of one European put option plus one share. These two portfolios are both worth $\max [S_T, X]$ at the expiration of the options. These options are European so they cannot be exercised prior to the expiration date. Clearly both portfolios must therefore have the same value today. The put-call parity is then

$$P_c + Xe^{-rT} = P_p + S_0.$$ 

When pricing European options at a time prior to maturity the price is equal to the expectation value of the payoff under the risk-neutral measurement $Q$ discounted with a constant rate $r$.

$$P_c(t) = e^{-r(T-t)} E^Q[\max[S_T - X, 0]|S_t] \quad (2.6)$$

To compute the price $P_c(t)$ Black-Scholes formula is used. For a European call option on a non-dividend-paying stock the Black-Scholes price is

$$P_c(t) = S_t N(d_1) - Xe^{-r(T-t)} N(d_2). \quad (2.7)$$

with

$$d_1 = \frac{\ln(S_t/X) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}},$$
$$d_2 = d_1 - \sigma \sqrt{T-t}. \quad (2.8)$$

The Black-Scholes price of an option depends on four different variables. The risk-free interest rate $r$, the stock price $S$, the remaining time to exercise $T-t$ and the stock price volatility $\sigma$. The strike price $X$ is a contract specific parameter. All variables except for the volatility can be observed on the market. When deciding which volatility to use in the pricing the situation is more complicated. It is possible to calculate historical volatility on historical stock price data but we need to know what the volatility will be from today and until the maturity date. To use the historical volatility is often a good approximation when one does not have market data on the option price.

It is possible to look at this problem in another way, the price of the option can be observed on the market. It is then possible to calculate which volatility that will
give the same theoretical option price as the market price, this is called the implied volatility. The implied volatility represents future market expectations.

To decide the volatility that should be used in convertible bond pricing it is suitable to use the implied volatility. When calculating implied volatility for a specific stock, prices for options traded on frequently basis are needed. In this thesis it is desired to use stock options with the same maturity as the convertible bonds. Since convertible bonds often have a time to maturity longer than one year and frequently traded stock options have time to maturity less than 6 months, the implied volatility of the stock options with the longest possible time to maturity will best show the market expectation of the volatility in the long run. The mid-spread is considered to be the market price of an option, midspread is the mean of the buy and sell price. It is not possible to express the volatility as a function of the other Black-Scholes variables and the option price. An iterative method has to be used to decide the implied volatility.

2.5 The Greek letters

The Greek letters or simply the Greeks are measures on the risk in a derivative. They are partial derivate of the value of the derivative w.r.t. one of the variables i.e. $S, t$ etc. The aim of a trader is to manage the Greeks so that all risks in a contract are acceptable. The more usual Greeks has a Greek name like delta, gamma etc. In this thesis there will be some unusual Greeks and they will not be given a name, just stated.

The Greek letter $\Delta$ delta is the change in the contract value $f$ according to a small change in asset price when everything else remains the same. The Greek letter $\Gamma$ gamma is the second partial derivate of the contract value w.r.t. the asset price.

$$\Delta = \frac{\partial f}{\partial S} \quad \Gamma = \frac{\partial^2 f}{\partial S^2}$$

If gamma is small, delta changes slowly. A large gamma in absolute terms indicates that delta is highly sensitive to the price of the underlying asset.

The Greek letter $\Theta$ theta is the rate of change of the value w.r.t. time when all else remains the same i.e.

$$\Theta = \frac{\partial f}{\partial t}.$$ 

The Greek letter $V$ vega is the change of the value w.r.t. a change in the volatility of the asset $\sigma$ i.e.

$$V = \frac{\partial f}{\partial \sigma}.$$
A high absolute vega indicates that the contract value is very sensitive to small changes in volatility.

For a convertible bond with stochastic interest rate there is also a matter of interest to study the influence of the first and second partial derivative w.r.t. \( r \) i.e.

\[
\frac{\partial f}{\partial r}, \quad \frac{\partial^2 f}{\partial r^2}.
\]

It will reveal the sensitivity of the convertible bond price toward changes in the interest rate.

### 2.6 Interest rate term structure

The basic principle of interest rate term structure is that the interest rate depends on the length of the loan. Normally a loan with a longer time to maturity has a higher interest than a loan with a shorter time to maturity because of the higher risk for the lender.

There are three equivalent ways to fully describe the interest rate term structure in the market. Suppose there is a zero-coupon bond with every possible time to maturity and face value of 1 SEK.

#### The discount function

The discount function \( d(t, T) \) describes the price at \( t \) of a zero-coupon bond with maturity at time \( T \).

#### Spot rate function

The spot rate \( i(t, T) \) is the interest to be paid at time \( t \) on a loan with \( T \) years to maturity. With the yields of our infinite set of zero-coupon bonds at time \( t \) it is possible to construct the spot rate function \( i(t, T) \).

#### Instantaneous forward rate

The instantaneous forward rate \( f(t, T) \) is the expected short rate at a time \( T \) decided at \( t \).

With any of these three it is possible to derive one of the others as well as the expected interest rates in the future.

The discount function \( d(t, T) \) describes the discount rate between to times where \( t \) may be a time in the future.

\[
\begin{align*}
    i(t, T) &= \frac{\int_{\tau=t}^{T} f(t, \tau) d\tau}{T - t} \\
    d(t, T) &= e^{-i(t; T) (T-t)}
\end{align*}
\]  
(2.9)
In the real market there is not an infinite number of zero-coupon bonds. By using market information on existing zero-coupon bonds it is only possible to determine a few points in the discount function or spot rate function.

Another drawback is that zero-coupon bonds seldom have a time to maturity longer than one year. To get information about the term structure for times more than a year from the present day it is necessary to use market data on coupon bonds. A coupon bond can be seen as a portfolio of zero-coupon bonds of different maturities. The yield for a coupon bond is not the same as the spot rate for the maturity time. Since a coupon bond has payments at several different occasions the yield can be described as a complicated average of the spot rates at all payment times. Using the yield of a coupon bond will therefore not give the exact information about the spot rate at any time. To define the theoretical term structure a model is proposed and parameters are chosen to fit the model as closely as possible to market data.

### 2.7 Bonds and interest rates

A zero-coupon bond, $ZCB$, with maturity $T$ is a contract which guarantees the holder that a principal value will be paid on the date $T$. Principal value is often called face value. The price at time $t$ of a bond maturing at $T$ is denoted as $p(t, T)$. Often it is convenient to set the principal value to 1. Note that the condition that $p(t, t) = 1$ is necessary to avoid arbitrage.

Bonds can also have coupons. Coupon bonds will give the owner a payment stream known as coupons, during the period $[0, T]$ at a regular interval e.g. once a year. These bonds provide the owner with a deterministic cash flow.

In Sweden the government issues bonds yearly with a maturity ranging from 5 to 10 years. The bond market in Sweden consists of bonds with different maturity as well as different maturity months. Looking at the Swedish government bond market there is approximately a coupon expiring every month.

Various short rate notations are used on the market, they can be used for either discrete or continuous compounding. In this thesis the main rates that will be used are the instantaneous forward rate and the short rate.

The instantaneous forward rate process $f(t, T)$ has the economic interpretation that it is the risk less rate of return on an investment over the infinitesimal interval $[T, T + dT]$ when the contract is made at time $t$. The short rate $r(T)$ is the risk less rate of return over $[T, T + dt]$ if the contract is made at $T$.

The instantaneous forward rate and the short rate is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T}$$

$$r(t) = f(t, t).$$  \hspace{1cm} (2.10)
The price of a zero coupon bond can now be expressed, between \( t \leq s \leq T \), as

\[
p_{ZCB}(t, T) = \exp\left(-\int_t^T f(t, s)\,ds\right).
\] (2.11)

On the Swedish market there are no zero-coupon bonds traded actively with maturity longer than one year. The bonds in Sweden are fixed coupon bonds that at some intermediate points in time provides a deterministic payment (coupon) to the holder of the bond. The coupons are normally a percentage of the face value. A coupon bond can be expressed as a collection of various zero-coupon bonds with different maturities. The price for a coupon bond \( p_{CB}(t, T_n) \) with maturity \( T_n \), face value \( N \) and coupons \( K_i \) at time interval \( i = 1, \ldots, n \) can be expressed as

\[
p_{CB}(t, T_1, \ldots, T_n) = N \cdot p(t, T_n) + \sum_{i=1}^n (K_i \cdot p(t, T_i)).
\]

In a continuous time, discounted with the spot rate, see equation 2.9, which varies over time, the price of the coupon bond is

\[
p_{CB}(t, T_1, \ldots, T_n) = N \cdot e^{-\int_t^{T_n} T_n} + \sum_{i=1}^n (K_i \cdot e^{-\int_t^{T_i} T_i}).
\]

A common use on the market is the concept yield to maturity \( y \), which can be explained as the fixed rate that gives the bond its market price. Face value can for simplicity be assumed to be 1.

\[
p_{CB}(t, T_1, \ldots, T_n) = \sum_{i=1}^n K_i \cdot e^{-y(T_i)}
\]

It is common to report in the daily news the bond’s price as its yield to maturity.

Another useful market term is duration. It can be viewed as a weighted average of the coupons and the face value. It can in some sense be seen as the “mean time to payment”. A large duration implies that the mean payment is far ahead in time. The unit for duration is time so a natural limit for the duration is \( 0 \leq D \leq T_n \).

\[
D = \frac{\sum_{i=1}^n T_i K_i e^{-yT_i}}{p}
\]

A bond with high duration is considered to be exposed to a higher risk than a bond with lower duration, since the “mean time to payment” is further away for bonds with high duration.


\section*{2.8 Short rate models}

When using Black’s model, see Hull \cite{3}, on interest rate derivatives the short rate is considered to be deterministic or even constant. Although being flexible and straightforward to use the fact that not allowing the rate to be stochastic is a major drawback for the model.

The use of a stochastic short rate has therefore been developed for the use of a more general approach. In a one-factor equilibrium model, the process for \( r(t) \) involves one source of uncertainty. The short rate process can be described as an Itô process on the form

\[ dr = \mu(r, t)dt + \sigma(r, t)dW_t. \]  

\text{(2.12)}

The second part of the equation uses a standard Wiener process as in section 2.2 rendering \( r(t) \) a stochastic behaviour.

There is today a lot of different short rate models but this thesis will only consider two cases, the \textit{Hull – White} and the \textit{Ho – Lee} models.

The Ho-Lee model is just a simplification of Hull-White. Both models have analytical tractability towards the initial forward rate curve but Hull-White also has a mean reverting term \( a \) that pushes the solution toward the mean i.e. the forward rate curve. Both short rate models are stated under the risk neutral measure \( Q \).

\begin{align*}
\text{Hull-White} & \quad dr = \left( \theta(t) - ar \right)dt + \sigma_{HW}dW_t \\
\text{Ho-Lee} & \quad dr = \phi(t)dt + \sigma_{HL}dW_t
\end{align*}

\text{(2.13)} \quad \text{(2.14)}

The term \( \sigma \) is the volatility of the rate. The mean reverting term \( a \) tends to pull the interest rate to some long run average level over time. When \( r \) is high mean reversion tends to cause it to have a negative drift, and vice versa for low \( r \). The economic interpretation is that when the interest rate is high, the economy tends to slow down and rates decline. The Ho-Lee model is a particular case of the Hull-White model with \( a = 0 \).

In this thesis the derivation of the Hull-White will be made and for the Ho-Lee the result will just be stated.

With knowledge in stochastic calculus, see Kloeden, Platen \cite{9}, \( r(t) \) has the following solution in the Hull-White model

\[ r(t) = r(0)e^{-at} + \int_0^t e^{-a(t-s)}\theta(s)ds + \int_0^t e^{-a(t-s)}\sigma dW_s. \]

The task now is to construct a bond using this stochastic rate. The price of a bond paying no coupons at time \( t \) with maturity \( T \) is \( p(t, T) \). Now assuming that \( p(t, T) \)
has what is called affine term structure, see Björk [4] the bond price expression becomes \( p(t, T) = e^{A(t, T) - B(t, T)r(t)} \). Applying Itô’s formula on this expression and using the Hull-White expression yields that

\[
\frac{\partial B(t, T)}{\partial t} - \left(1 - \frac{\partial B(t, T)}{\partial t}\right) r(t) + \left(\theta(t) - ar(t)\right) B(t, T) + \frac{1}{2}\sigma^2 B^2(t, T) = 0.
\]

Collecting all terms that excludes and includes \( r(t) \) and considering the boundary condition \( p(T, T) = 1 \) which is necessary for the absence of arbitrage ends up in solving these two equations

\[
\begin{cases}
\frac{\partial}{\partial t} B(t, T) = a B(t, T) - 1 \\
B(T, T) = 0
\end{cases}
\]

\[
\begin{cases}
\frac{\partial}{\partial t} A(t, T) = \theta(t) B(t, T) - \frac{1}{2}\sigma^2 B^2(t, T) \\
A(T, T) = 0
\end{cases}
\]

First solving \( B(t, T) \) and using that to solve \( A(t, T) \) leads to

\[
B(t, T) = \frac{1}{a} \left(1 - e^{-a(T-t)}\right)
\]

\[
A(t, T) = \int_t^T \left(\frac{1}{2}\sigma^2 B^2(s, T) - \theta(s) B(s, T)\right) ds.
\]

(2.15)

The function \( \theta(t) \) is still not known. When solving \( \theta(t) \) there has to be connection between the forward rate and the affine model. \( \theta(t) \) is chosen in order to fit the theoretical bond \( P(0, T) \) to the observed bond price \( P^*(0, T) \). Using the relation between bond price and forward rate, \( f^*(t, T) = -\partial \left(\log P^*(t, T)\right)/\partial T \) will make solving easier.

For the affine model the forward rate for the time interval \([0,T]\) is given by \( f(0, T) = B_T(0, T)r(0) - A_T(0, T) \).

Using this equation with the functions \( A(t, T) \) and \( B(t, T) \) from equation 2.15 one gets

\[
f(0, T) = e^{-aT} r(0) - \frac{\sigma^2}{2a^2} \left(1 - e^{-aT}\right)^2 + \int_0^T \theta(s)e^{-a(T-s)} ds.
\]

(2.16)

Now the theoretical forward rate \( f(0, T) \) can be observed on the market as \( f^*(0, T) \). So finding \( \theta(t) \) can now be reduced to solving the following equations

\[
f^*(0, T) = x(T) - g(T),
\]

\[
g(t) = \frac{\sigma^2}{2a^2} \left(1 - e^{-at}\right)^2,
\]

\[
\dot{x} = -ax(t) + \theta(t) \quad \text{where } x(0) = r(0).
\]
It now turns out that this can be solved and \( \theta(t) \) will have both a market forward rate term and one derivative of it.

\[
\theta(t) = \frac{\partial}{\partial t} f^*(0, t) + af^*(0, t) + \frac{\sigma^2}{2a} \left( 1 - e^{-2at} \right)
\] (2.17)

Choosing \( \theta(t) \) as above will produce the desirable affine term structure for the bond price such that \( p(0, T) = p^*(0, T) \) for all \( T \geq 0 \).

The bond prices are log normal so by using Black-Scholes formula the short rate \( r(t) \) and the bond price \( p(t, T) \) can, for the Hull-White model, be expressed as

\[
r(t) = f^*(0, t) + \frac{\sigma^2}{2a^2} (1 - e^{-at})^2 + \sigma \int_0^t e^{-a(t-s)} dW_s,
\] (2.18)

\[
p(t, T) = \frac{p^*(0, T)}{p^*(0, t)} \exp \left( B(t, T) f^*(0, t) + \frac{\sigma^2}{4a} B^2(t, T) (1 - e^{-2at}) - B(t, T) r(t) \right).
\] (2.19)

**Bond options**

In Sweden bond options are traded in the over-the-counter (OTC) market, usually between two financial institutions or between a financial institution and one of its clients. It can in some sense be seen as tailored by a financial institution to meet a specific need of a client. A bond option is an option to buy or sell a particular bond at a certain date for a particular price or yield.

The work by Brigo and Mercurio [1] explains Jamshidian’s decomposition theorem valid for short rate models. It states that an option on a coupon bond can be explicitly priced as a collection of zero-coupon options,

\[
CBO(t, T, K, X) = \sum_{i=1}^n K_i ZBO(t, T_i, X_i),
\]

where \( CBO \) is the coupon bond option and \( ZBO \) are the zero-coupon bond options. So when pricing existing bond options it is easier to decompose the actual bond into several zero-coupon bond options and treat them one by one.

When using a short rate model such as Hull-White one has to calibrate the model i.e. estimate \( a \) and \( \sigma \). This can be done by observing interest rate derivative products such as bond options that are priced on the market. The bond prices \( p^*(0, t) \) and \( p^*(0, T) \) can be directly observed on the market. Those bond prices can then be used to price bond options, both callable and putable.

By defining

\[
S, T : \text{Maturity for option and for zero-coupon bond} \\
K : \text{face value of bond} \\
X : \text{strike price for the option} \\
P(0, t) : \text{price at } t \text{ for a zero-coupon bond with face value } 1
\]
expressions for both call and put bond options with Hull-White as a short rate model will be

\[
\begin{align*}
\text{call option} &= KP(0,T)N(h) - XP(0,S)N(h - \sigma_p), \\
\text{put option} &= XP(0,S)N(-h + \sigma_p) - KP(0,T)N(-h). \quad (2.20)
\end{align*}
\]

where

\[
\begin{align*}
h &= \frac{1}{\sigma_r} \ln \left( \frac{KP(0,T)}{XP(0,S)} \right) + \frac{\sigma_r}{2}, \\
\sigma_r &= \frac{\sigma}{a} (1 - e^{-a(T-S)}) \sqrt{(1 - e^{-2aS})/2a}.
\end{align*}
\]

So observing market pricing of bond options will give an estimate of the mean reversion term \(a\) and the volatility \(\sigma\) to be used in the model to calculate other contracts with different conditions.

Using the same procedure for the Ho-Lee model equation 2.14 yields the following result

\[
\begin{align*}
\phi(t) &= \frac{\partial}{\partial t} f^*(0, t) + \sigma^2 t \\
r(t) &= f^*(0, t) + \frac{\sigma^2}{2} t^2 + \sigma W_t, \\
p(t, T) &= p^*(0, T) \exp \left( (T-t)f^*(0, t) - \frac{\sigma^2}{2} t(t-T)^2 - (T-t)r(t) \right)
\end{align*}
\]

The put and call bond options when using Ho-Lee will be as in equation 2.20 with the exceptions that \(\sigma_r = \sigma(T-S)\sqrt{\mathcal{S}}\) and \(h\) will include bond prices calculated as above.
Chapter 3

Convertible bonds

3.1 Introduction to Convertible Bonds

A convertible bond is a bond that may be converted into stocks. A convertible bond is normally issued by a company. Just like a normal bond it has a face value that will be paid back at the date of maturity. The holder has the right to convert the bond into a predetermined number of shares in the company in a given time period. The choice to convert or not depends on the stock price and the interest rate, therefore the convertible bond price also depends on both the stock price and the interest rate which makes pricing more complex than bond and stock option pricing. The following notation will frequently be used in the thesis.

\[ N \] : Face value of convertible bond
\[ n \] : Conversion ratio, stocks per convertible bond
\[ X \] : Conversion rate, \( X \equiv N/n \) (3.1)

In a simplified case the value of the convertible bond equals \( \max(nS, N) \) at the date of maturity. The holder chooses either to convert into stocks or receive the face value of the bond depending on which part is worth the most. The stocks are preferable if the stock price \( S \) is greater then \( X \), i.e. \( S > X \).

3.2 Issuing convertible bonds

There are mainly two different reasons for a company to issue convertible bonds. They can be issued to employees as a bonus or to motivate them to work harder. The advantage of issuing convertible bonds instead of stock options is that convertible bonds have a guaranteed payback unless to company goes bankrupt.

For a company in need of financing issuing new shares or convertible bonds to shareholders or other investors is a way to raise money. When issuing new shares the total number of stocks in the company will increase and the company’s profit per stock will decrease, this is called dilution. In this case issuing convertible bonds has some advantages since the dilution effect is delayed.
3.3 Contract conditions

The market for convertible bonds is not a standardized market. Conditions for convertible bonds are usually very different from case to case.

One feature is the date of maturity, the day when the face value of the bond will be returned if the bond has not already been converted. The period of conversion is of great importance. There are no convertible bonds traded on the market where conversion is possible in a single day only, like a European contract. Convertible bonds that may be converted at any time like an American contract is not a common practice either. Usually convertible bonds can be converted in a predetermined period of time with a length of 2 months up to several years. Some convertible bonds may be of a Bermudan type when conversion is possible on a discrete number of occasions. It is important to notice that there is always some time between the last day of conversion and the maturity date of the bond. That makes the conversion decision a little more complicated because the face value will be received at a later time.

Convertible bonds usually have coupon payments just like ordinary bonds. The holder of the bond then receives a coupon payment from the issuing company. There may be different number of coupon payments per year but one coupon per year is a standard on the Swedish market. The coupon interest can be determined in advance or may be linked to a floating market rate.

Some convertible bond data specific for each contract are the face value of the bond, the conversion value and the conversion ratio.

3.4 Basic convertible bond pricing

To price a convertible bond contract a lot of variables have to be taken into consideration. Besides the specific contract conditions the price depend on the stock price, stock dividends, date of dividend, stock volatility, interest rate, interest rate volatility and interest and stock price correlation.

The price of the convertible bond has a different behaviour depending on the stock price. When stock price tends to zero the convertible bond price tends to the price of the bond part. When stock price tends to infinity the convertible bond price is just stock price times conversion rate i.e.

\[
S \to 0 \implies CB \to Bond
\]

\[
S \to \infty \implies CB \to nS.
\]

When the stock price tends to zero there is no chance that conversion will be optimal and therefore the convertible bond will be equivalent to an ordinary bond. When the stock price \( S \) is much larger than \( X \) the probability of conversion tends to 1 and therefore the convertible bond is equivalent to a normal stock times the conversion ratio.
The price of the convertible has to be consistent with market behaviour so the following theorem is always valid.

**Theorem:** *The price of a convertible bond is always equal or greater then nS.*

**Proof:** Suppose the market is free of arbitrage and there are no transaction costs. Consider a convertible bond that has the price $CB < nS$. Then an investor could short sell n stocks at stock price $S$ and buy a convertible bond at price $CB$. The investor would then convert the convertible bond into n stocks and return the stocks that had been short sold. The profit of these transactions would then be $nS - CB > 0$, which is an arbitrage.

This works even if the convertible bond is bought in its European period, but the investor would have to wait until the convertible bond could be converted before the short sold stocks could be returned. Since the market is free of arbitrage the situation where $CB < nS$ can never occur. Hence the price of the CB is always at least the conversion value, that gives the constraint $CB \geq nS$.

The price of the CB is also always at least the “straight value”, the value of the corporate bond without the conversion feature. Early conversion of the convertible bond is never optimal in the absence of dividends, see Wilmott p. 465 [5]. Since $CB \geq nS$ it is better to sell the convertible bond than to convert it. In a situation where the stock has a dividend, early conversion may be optimal. The aspect of early conversion will be reviewed further in section 4.3. A discrete dividend $D$ is paid at time $t_d$, let $t_d^-$ be just prior to when the dividend is paid and $t_d^+$ just after. At time $t_d$ the stock price will fall the same amount as the dividend,

$$S(t_d^+) = S(t_d^-) - D.$$  

When the convertible bond is in its European period the value of the convertible bond has to be continuous in time

$$CB(S(t_d^-), t_d^-) = CB(S(t_d^+), t_d^+),$$

so

$$CB(S(t_d^-), t_d^-) = CB(S(t_d^-) - D, t_d^+).$$

When the stock has a dividend and the convertible bond is in its American period it is possible that the jump in stock price will cause a jump in the convertible bond price, that is

$$CB(S(t_d^-), t_d^-) = \max \left[ CB(S(t_d^+), t_d^+), nS(t_d^-) \right].$$

**Pricing using deterministic short rate**

On the last day of conversion the convertible bond price can be stated as

$$CB(\tau) = \max \left[ nS(\tau), (N + K)e^{-\tau(T-\tau)} \right]. \quad (3.2)$$
A coupon of size $K$ is normally paid on the same day as the face value $N$. The conversion will be made if

$$S(\tau) > (N + K)e^{-i(\tau,T)(T-\tau)}/n = (X + K/n)e^{-i(\tau,T)(T-\tau)}.$$ 

Let

$$Z = (X + K/n)e^{-i(\tau,T)(T-\tau)}.$$ 

Then

$$CB(\tau) = \max \left[ nS(\tau), nZ \right] = n \max \left[ S(\tau), Z \right]$$

$$= nZ + n \max \left[ S(\tau) - Z, 0 \right]$$

$$= (N + K)e^{-i(\tau,T)(T-\tau)} + n \max \left[ S(\tau) - Z, 0 \right]. \quad (3.3)$$

A very useful equivalence can be made when there is no reason for early conversion of the convertible bond.

The convertible bond can be considered as made of two parts, a bond part and a call option part. Then using deterministic short rate the convertible bond price at an earlier time $t < \tau$ can be expressed as the present value of the bond part and a discounted expectation value under the risk free measure $Q$ of the call option part.

$$CB(t) = (N + K)e^{-i(t,T)(T-t)} + ne^{-i(t,\tau)(\tau-t)}E^Q[\max(S(\tau) - Z, 0)] \quad (3.4)$$

So the convertible bond in this case is equivalent to a bond plus a European call option on the stock. If the last day of conversion and maturity date of the bond would coincide the strike of the option would be the face value of the bond.

**Pricing using stochastic short rate**

When there is no reason for early conversion an analytical solution for a convertible bond with stochastic interest rate can be obtained. Since the decision at time $\tau$ is either to keep the bond or convert to $n$ stocks the payoff of a convertible bond at time $\tau$ is then

$$\max \left[ nS(\tau), p(\tau,T) \right],$$

where $p(\tau,T)$ is zero-coupon bond of the same risk class as the stock maturing at time $T$.

One can now use the zero-coupon bond as the numeraire, see Benninga, Björk, Wiener [11], and express the price of the convertible bond as

$$CB(t) = p(t,T)E^T_t \left[ \max \left[ Z_\tau, 1 \right] \right],$$

where $E^T_t$ denotes the expectation under the “forward neutral” measure $Q^T$ with the bond as a numeraire. The process $Z$ is defined by

$$Z_t = \frac{nS(t)}{p(t,T)}.$$
Simplifying the price gives that
\[ CB(t) = p(t, T) + p(t, T)E_t^T \max \left[ Z_t - 1, 0 \right] \]. \hspace{1cm} (3.5)

The dynamic of the \( Z \) process is thus given by
\[ dZ_t = Z_t \sigma_Z(t) dW(t) \]
where \( \sigma_Z = \sigma_{Stock} + \sigma_{Bond}(t, T) \). The volatility for the bond is the bond price volatility as defined in equation 5.3 and observe that the \( Z \) process has no drift. Under the assumption that \( \sigma_Z \) is deterministic \( Z \) has a lognormal distribution and a variation of Black-Scholes formula can be used to obtain the price as
\[ CB(t) = p(t, T) + nS(t)N[d_1] - p(t, T)N[d_2], \hspace{1cm} (3.6) \]
where
\[ d_1 = \frac{1}{\sqrt{\sigma^2(t, \tau)}} \left( \ln \left( \frac{nS(t)}{p(t, T)} \right) + \frac{1}{2} \sigma^2(t, \tau) \right), \]
\[ d_2 = d_1 - \sqrt{\sigma^2(t, \tau)}. \]
\[ \sigma^2(t, \tau) = \int_t^\tau \| \sigma_Z(u) \|^2 du. \]

A convertible bond may have a call feature. That means that the issuing company has the right to purchase back the convertible bond anytime (or at a specified period) for a specified amount. If the issuing company has the right to repurchase the convertible bond for an amount \( M_C \) then the call feature will have the following effect on convertible bond pricing
\[ CB(S, t) \leq M_C. \]

The CB price cannot be higher than \( M_C \) otherwise it would be an advantage for the company to repurchase them at \( M_C \).

A convertible bond may also have a put feature. That gives the holder of the CB the right to the CB to the company for a specified amount \( M_P \)
\[ CB(S, t) \geq M_P. \]

Both \( M_C \) and \( M_P \) can be time dependent. Swedish convertible bonds rarely have a put or call feature.

### 3.5 Default risk

There is a risk that the company issuing the convertible bond will go bankrupt. In that case the holder of the convertible bond will not receive any coupon payments
and the face value of the bond will not returned. The right to convert into stocks will also be worthless since the stock price will be close to zero. In the case where default risk is taken into account \( CB \rightarrow 0 \) as \( S \rightarrow 0 \). As a result of the default risk the interest rate used to discount a company bond will be greater than the risk-free interest. How much greater the interest rate will be depends on how risky the company is. There is a risk that a company issuing convertible bonds go bankrupt just because they do not have the money to pay back the face value of the bond to the convertible bond holders.

Consider two zero-coupon bonds both with time to maturity \( T - t \) and face value \( N \). One bond is issued by the government and has a risk-free interest rate, its yield is \( y_1 \). The other bond is issued by a company and therefore has a risky interest rate, its yield is \( y_2 \). Since the company bond is more risky \( y_2 \geq y_1 \).

The prices of the bonds can be described as

\[
\begin{align*}
\text{Government bond} & : \quad p_G(t,T) = Ne^{-y_1(T-t)}, \\
\text{Company bond} & : \quad p_C(t,T) = Ne^{-y_2(T-t)} \\
p_C(t,T) & \leq p_G(t,T).
\end{align*}
\]

There is a risk that the company bond will default. Consider the probability of default to be \( \alpha \) and \( 0 \leq \alpha \leq 1 \), then \( E[p_C(T,T)] = (1-\alpha)N \). The expected rate of return between time \( t \) and \( T \) is equal for both bonds namely \( e^{y_1(T-t)} - 1 \). At time \( t \) the bond prices are known as \( p_G(t,T) = Ne^{-y_1(T-t)} \) and \( p_C(t,T) = Ne^{-y_2(T-t)} \).

This gives a connection between the risky interest rate and the probability of default.

\[
E\left[\frac{p_G(T,T)}{p_G(t,T)}\right] = E\left[\frac{p_C(T,T)}{p_C(t,T)}\right] = e^{y_1(T-t)}
\]

\[
\Rightarrow e^{y_1(T-t)} = \frac{(1-\alpha)N}{Ne^{-y_2(T-t)}}
\]

\[
\Rightarrow y_2 = y_1 - \frac{ln(1-\alpha)}{T-t} \quad (3.7)
\]

Since \( -ln(1-\alpha) \) is positive \( y_2 \geq y_1 \). Obviously the risky interest rate will increase with increasing default risk.

One can observe the differences between companies and government interest rates by constructing spot yield curves using bond data from the market.

Looking at Figure 3.1 one can clearly see differences in the curves, the government yield curve being the lowest and the others above reflecting the default risk. The differences between the curves in the graph is called the credit spread. In this thesis the credit spread observed on the market will be used to treat the default risk. There are ways to deal with credit risk in a more thorough way however that will not be done in this thesis.
3.6 Deriving the PDE for convertible bond

When interest rates are stochastic a convertible bond has a value of the form

$$ V = V(S, r, t) $$

The value of the convertible bond is now a function of both $S$ and $r$. The asset price and the interest rate are under the objective measure $P$ modelled by

$$ dS = \mu S dt + \sigma S dW_1 $$
$$ dr = \gamma dt + \omega dW_2 $$

Using Itô’s formula on $V(S, r, t)$ and the fact that $dW_1 \cdot dW_2 = \rho dt$ yields

$$ dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial r} dr + \frac{1}{2} \left( \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \omega^2 \frac{\partial^2 V}{\partial r^2} + 2 \rho \sigma \omega S \frac{\partial^2 V}{\partial r \partial S} \right) dt. $$

Consider a portfolio consisting of one sold European convertible bond with maturity $T_1$, $\Delta_2$ zero coupon bonds with maturity $T_2$, $\Delta_1$ of the underlying asset and $\Lambda$ of the risk-free paper $B$. The portfolio $\Pi$ is thus self-financing.

$$ \Pi = -V + \Delta_2 ZCB + \Delta_1 S + \Lambda B $$

Differentiating $\Pi$ yields that

$$ d\Pi = -dV + \Delta_2 dZCB + \Delta_1 dS + \Lambda dB. $$
The risk-free paper $B$ as explained in section 2.3 follows $dB = rBdt$. By choosing $\Delta_1$ and $\Delta_2$ so that the randomness in $d\Pi$ is eliminated it turns out that choosing

$$
\Delta_1 = \frac{\partial V}{\partial S}, \\
\Delta_2 = \frac{\partial V}{\partial r} / \frac{\partial ZCB}{\partial r},
$$

makes the portfolio risk free.

In a no-arbitrage market the portfolio increment should be the same as investing the portfolio value earning the risk free rate $r$ i.e. $d\Pi = r\Pi dt$.

Terms involving $T_1$ and $T_2$ can now be grouped together separately and dropping the subscripts and the $dt$ term yields

$$
\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \gamma \frac{\partial V}{\partial r} +
+ \frac{1}{2} \left( \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \omega^2 \frac{\partial^2 V}{\partial r^2} + 2\rho\sigma\omega S \frac{\partial^2 V}{\partial r \partial S} \right) - rV = 0 \tag{3.8}
$$

This is the PDE for a convertible bond with stochastic interest rate and asset. The right side $= 0$ indicates that the bond is a European bond, in the American case the right side is $\leq 0$. For the solution to be unique a boundary condition is necessary. The price of a convertible bond at time $\tau$ is $V(S, \tau) = (N + K)e^{-i(tT)}(T-\tau) + \max[S(\tau) - Z, 0]$, see equation 3.3.

Under the risk neutral measure $Q$ the short rate term $\gamma$ will be different, see equation 2.13.
Chapter 4

Numerical methods

When pricing convertible bonds there is an explicit expression for the price at the end of the conversion period $\tau$. The price at time $\tau$ is just $V(S, \tau) = \max[nS(\tau), (N + K)e^{-r(\tau,T)}(T-\tau)]$, where $nS$ is the conversion rate times the stock price and $(N + K)$ equals the face value plus coupon. Finite-difference methods will be used for the one-factor model as well as for the two-factor model following a scheme from Wilmott ch. 36 [5].

4.1 One-factor model

In the one-factor model the interest rate is considered fixed. The value of a convertible bond at each grid point is

$$V_i^k = V(i \delta S, \tau - k \delta t),$$

$$S = i \delta S,$$

$$t = \tau - k \delta t,$$

where $0 \leq i \leq I$ and $0 \leq k \leq K$. The direction of time is reversed, when $k$ increases real time decreases. Since the pricing is under the risk neutral measure $Q$ the stock price process is modelled as a geometric Brownian motion $dS = rSdt + \sigma SdW$.

By using Itô’s formula on $V(S, t)$ and making a risk neutral portfolio one gets the following PDE expression

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \tag{4.1}$$

The expression above is for a European contract, for an American contract the right side becomes $\leq 0$.

Using Crank-Nicolson method, central differences for partial derivatives and symmetric central differences for second partial differences the equation becomes

$$\frac{V_i^k - V_i^{k+1}}{\Delta t} + \frac{a_i^{k+1}}{2} \left( \frac{V_{i+1}^{k+1} - 2V_i^{k+1} + V_{i-1}^{k+1}}{\Delta S^2} \right) + \frac{a_i^k}{2} \left( \frac{V_{i+1}^k - 2V_i^k + V_{i-1}^k}{\Delta S^2} \right) +$$
\begin{align*}
\frac{b_l^{k+1}}{2} \left( \frac{V_{i+1}^{k+1} - V_{i-1}^{k+1}}{2\Delta S} \right) + \frac{b_l^{k}}{2} \left( \frac{V_{i+1}^{k} - V_{i-1}^{k}}{2\Delta S} \right) + \frac{c_{i}^{k+1}}{2} V_{i}^{k+1} + \frac{c_{i}^{k}}{2} V_{i}^{k} &= O(\Delta t + (\Delta S)^2).
\end{align*}

This expression can be arranged so that \((k+1)\)-terms are on the left and \(k\)-terms on the right. All coefficients can, at each grid point, be expressed as \(A_i^k, B_i^k, C_i^k\).

Doing that leads to the following

\begin{align*}
V_{i-1}^{k+1}(-A_i^{k+1}) + V_{i}^{k+1}(B_i^{k+1} - 1) + V_{i+1}^{k+1}(-C_i^{k+1}) &= V_{i-1}^{k+1}(A_i^{k}) + V_{i}^{k+1}(-B_i^{k} - 1) + V_{i+1}^{k+1}(C_i^{k})
\end{align*}

The Crank-Nicolson method can now be written in matrix form. The matrices will be tridiagonal, the boundary conditions can be extracted from the matrices to separate vectors. This will reduce the matrices to square tridiagonal matrices.

The system can now be written as

\begin{align*}
M_L^{k+1} v^{k+1} + \tau^{k+1} = M_R^{k} v^{k} + p^{k}.
\end{align*}

The boundary condition for the convertible bond at time \(\tau\) is completely known as \(V(S, \tau) = \max[nS(\tau), (N + K) e^{r(T-\tau)}]\). For all \(t\), \(V(S, t) \sim nS; S \to \infty\) and when \(S \to 0\) the convertible bond price is just the price of the bond. This makes \(M_L^{k+1}, M_R^{k}, \tau^{k+1}\) and \(p^{k}\) completely known so solving the system from time \(\tau\) to 0 is just a matter of iterating.

Consider a convertible bond contract where early conversion is allowed only just prior to a stock dividend. Let \(\tau_d^-\) be just prior to when the dividend is paid and \(\tau_d^+\) just after the dividend. Since calculations are done in the reversed time order the convertible bond price at \(\tau_d^+\) will be calculated first. The convertible bond price at \(\tau_d^-\) can then be decided using

\begin{align*}
CB(S(\tau_d^-), \tau_d^-) = \max \left[ CB(S(\tau_d^+), \tau_d^+), nS(\tau_d^-) \right].
\end{align*}

After deciding the convertible bond price at \(\tau_d^-\) the calculations proceed in the normal way until the next dividend occurs.

This way of treating early conversion is used in both the one-factor model and the two-factor model.

### 4.2 Two-factor model

In the two-factor model the interest rate is now considered to be stochastic. The value of the convertible bond at each grid point is

\begin{align*}
V_{i,j}^k &= V(i \delta S, j \delta r, \tau - k \delta t) \\
S &= i \delta S \\
r &= j \delta r \\
t &= \tau - k \delta t
\end{align*}
where $0 \leq i \leq I$, $0 \leq j \leq J$ and $0 \leq k \leq K$. The direction of time is as before reversed, when $k$ increase real time decreases. Since the pricing is under the $Q$ measure the asset is modelled as before and the short-rate process is the Hull-White model $dr = (\theta(t) - ar)dt + \sigma dW_t$. By using Itô’s formula on $V(S, r, t)$ and making a risk neutral portfolio portfolio yields the following expression

$$
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \rho \sigma \omega \frac{\partial^2 V}{\partial S \partial r} + \frac{\omega^2}{2} \frac{\partial^2 V}{\partial r^2} + rS \frac{\partial V}{\partial S} + (\theta(t) - ar) \frac{\partial V}{\partial r} - rV = 0 \quad (4.3)
$$

Again this is a European contract, for an American contract the right side will be $\leq 0$.

By again using Crank-Nicolson method, central differences for partial derivatives and symmetric central differences for second partial differences the equation becomes

$$
\frac{V^{k+1}_{i,j} - V^{k}_{i,j}}{\Delta t} + \frac{a^{k+1}_{i,j}}{2} \left( \frac{V^{k+1}_{i+1,j} - 2V^{k+1}_{i,j} + V^{k+1}_{i-1,j}}{\Delta S^2} \right) + \frac{a^{k}_{i,j}}{2} \left( \frac{V^{k}_{i+1,j} - 2V^{k}_{i,j} + V^{k}_{i-1,j}}{\Delta S^2} \right) + 
\frac{\partial V^{k}_{i,j}}{\Delta S} \left( \frac{V^{k}_{i+1,j+1} - V^{k}_{i+1,j-1} - V^{k}_{i-1,j+1} + V^{k}_{i-1,j-1}}{4\Delta S \Delta r} \right) + 
\frac{\partial V^{k}_{i,j}}{\Delta r} \left( \frac{V^{k}_{i+1,j} - V^{k}_{i-1,j}}{2\Delta S} \right) + f^{k+1}_{i,j} V^{k+1}_{i,j} - f^{k}_{i,j} V^{k}_{i,j} = O(\Delta t + (\Delta S)^2 + (\Delta r)^2).
$$

As in the one-factor model this expression can be arranged in $(k + 1)$ and $k$ terms. The coefficients can, at each grid point, be expressed as $A^{k}_{i,j},\ldots,E^{k}_{i,j}$. This yields the following result

$$
V^{k+1}_{i+1,j+1}A^{k+1}_{i,j} + V^{k+1}_{i+1,j}B^{k+1}_{i,j} + V^{k+1}_{i+1,j-1}(-A^{k+1}_{i,j}) + V^{k+1}_{i,j+1}C^{k+1}_{i,j} + 
\frac{V^{k+1}_{i,j}}{2} \left( -1 - D^{k+1}_{i,j} + \frac{f^{k+1}_{i,j}}{2} \right) + V^{k+1}_{i,j+1}E^{k+1}_{i,j} + V^{k+1}_{i,j-1}(-A^{k+1}_{i,j}) + 
\frac{V^{k+1}_{i,j}}{2} \left( -1 - D^{k+1}_{i,j} + \frac{f^{k+1}_{i,j}}{2} \right) + V^{k+1}_{i,j+1}E^{k+1}_{i,j} - V^{k+1}_{i,j-1}(-A^{k+1}_{i,j}) + 
V^{k}_{i+1,j+1}(-A^{k}_{i,j}) + V^{k}_{i+1,j}(-B^{k}_{i,j}) + V^{k}_{i+1,j-1}A^{k}_{i,j} + V^{k}_{i,j+1}C^{k}_{i,j} + 
\frac{V^{k}_{i,j}}{2} \left( -1 - D^{k}_{i,j} + \frac{f^{k}_{i,j}}{2} \right) + V^{k}_{i,j+1}E^{k}_{i,j} + V^{k}_{i,j-1}(-A^{k}_{i,j}) + 
\frac{V^{k}_{i,j}}{2} \left( -1 - D^{k}_{i,j} + \frac{f^{k}_{i,j}}{2} \right) + V^{k}_{i,j+1}E^{k}_{i,j} - V^{k}_{i,j-1}(-A^{k}_{i,j}).
$$
This expression can now be written in matrix forms but they will not have the simple tridiagonal structure as in the one-factor model but a more complicated form. It can nevertheless be solved in a similar way by solving

\[ M_{k+1}^{L} v^{k+1} + r^{k+1} = M_{k}^{R} v^{k} + p^{k} \]  

(4.4)

The boundary conditions for \( S = 0 \) and \( S_{\text{max}} \) are the same as for the one-factor model. Deciding the boundary conditions for the lowest short rate \( r_{\text{min}} \) and the highest short rate \( r_{\text{max}} \) is more complicated. In reality \( r_{\text{min}} = 0 \) is a natural boundary, however \( r = 0 \) provides no special information about the convertible bond price, the exact solution is not known and Neumann boundary conditions cannot be used. Therefore \( r_{\text{min}} \) does not necessarily have to be equal to zero.

A known boundary condition is

\[ \lim_{r \to \infty} \frac{\partial V(S, r, t)}{\partial r} \to 0. \]

It can be difficult to use this boundary condition since for \( r_{\text{max}} \) to be big enough the number of gridpoints and computational time will be too big. Since the exact solution is not known for any \( r \) and Neumann conditions is not suitable, finding a good approximation to the solution is a way to set the boundaries.

In this thesis an approximative pricing method is used to set the \( r_{\text{min}} \) and \( r_{\text{max}} \) boundaries. In this approximation the convertible bond is treated as a bond plus an american call option. The bond price is decided using the short rate model as in section 2.8. The price of the american call option part is calculated using a one-factor model. The strike of the american call option is time dependent and equal to the expected bond price at each time.

### 4.3 The implemented program

The program constructed in this project is able to price convertible bonds with various contract conditions. The program is capable of handling discrete coupon payments and discrete stock dividends. All price calculations will take into consideration the possibility of early conversion. The contracts that will be priced will allow early conversion only just prior to the stock dividend. In reality early conversion is often allowed in a large timespan. It is probable but not proven that early conversion is optimal only just prior to a stock dividend. During the work with this thesis early conversion for American convertible bonds has been optimal only just prior to stock dividends for all contracts that have been examined.

Crank-Nicolson method has an order of \( \mathcal{O}(\Delta t + (\Delta S)^2 + (\Delta r)^2) \) so by letting \( E_1 = \Delta t + (\Delta S)^2 + (\Delta r)^2 \) and \( E_2 = \frac{\Delta t}{4} + \left(\frac{\Delta S}{2}\right)^2 + \left(\frac{\Delta r}{2}\right)^2 \) etc. and observing that \( E_{i+1} = \frac{E_i}{2} \) the errors can be computed and order of convergence testified. Using equation 3.6 as an exact solution for a European convertible bond and the
Table 4.1. Prices and errors with different $dS$, $dr$ and $dt$.

<table>
<thead>
<tr>
<th>Stepsize</th>
<th>$dS, dr, dt$</th>
<th>$dS/2, dr/2, dt/4$</th>
<th>$dS/4, dr/4, dt/16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>35.5541</td>
<td>35.5417</td>
<td>35.5387</td>
</tr>
<tr>
<td>Error</td>
<td>0.0166</td>
<td>0.0042</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

numerical solution an error is computed. Using a contract similar to the Telia contract in Table 6.1 but with no early conversion Table 4.1 shows the prices and the error for different step sizes, the exact price are 35.5375. It verifies that the order of convergence is correct for various step sizes for $dt$, $dS$ and $dr$.

It has been observed what effect the choice of $S_{\text{max}}$, $r_{\text{min}}$ and $r_{\text{max}}$ will have on the solution, thereafter suitable boundaries have been chosen. All tests have verified that the program performs correctly.
Chapter 5

Implementation

5.1 Implementation step by step

1. Decide the term structure of the interest rate using Nelson-Siegel’s parameterised function. Optimal parameters can be found by using market data for Swedish government and company bonds. The optimal parameters can be used to state the instantaneous forward rate function. The forward rate function and its derivative are needed to fully describe the stochastic short rate processes. A separate set of Nelson-Siegel parameters are optimized for each issuer of bonds.

2. The stochastic interest rate process for Hull-White includes the interest rate volatility $\sigma$ and mean reversion factor $a_{HW}$, Ho-Lee also includes a volatility. By pricing government bond options using Hull-White and Ho-Lee their parameters can be optimized by comparison to market data for these options. By observing historical behaviour on company interest rates and making some assumptions specific company parameters for Hull-White and Ho-Lee can be estimated.

3. Market data for stock options are used to estimate the implied volatility to be used in the stock price process.

4. Using the results from the previous steps convertible bond prices can be decided and different models can be compared.

5.2 Term structure models

When trying to decide the term structure using market data and an optimization process a parametric function is used to describe the term structure. Normally a parameterized function for the spot rate or the instantaneous forward rate is used. There are a number of steps to go through to derive the term structure.

1. Choose a function, $f(0, T; b)$ where $b$ is a set of model parameters, for the
instantaneous forward rate. There are many different parameter models to choose from. In this thesis Nelson-Siegel’s model will be used. Other models tend to be over-parameterized.

Nelson-Siegel’s Model:

\[ f(0, T; b) = \beta_0 + \beta_1 e^{-T/\tau} + \beta_2 \frac{T}{\tau} e^{-T/\tau} \]

\[ b = (\beta_0, \beta_1, \beta_2, \tau) \]  \hspace{1cm} (5.1)

2. Calculate the coupon bond prices \( P_k^M \) for every bond \( k \) using the parameterized instantaneous forward rate function.

\[ P_k^M = \sum_j c_j e^{-\int_{\tau_0}^{\tau_j} f(t, \tau) \, dt} \]

3. Let the weighted sum of pricing errors compared to market data be the objective function \( G \),

\[ G = \sum_k \omega_k (P_k^M - P_k)^2, \]

In Bolder, Streliski [10] a weighting is described where \( D \) is the duration of a bond

\[ \omega_k = \frac{1/D_k}{\sum_j 1/D_j} \]

The optimization model will minimize errors in the prices of the bonds. Errors in the interest rate will have more effect on prices for bonds with a long time to maturity since the discounting is during a longer time span. Without the weighting the method would over fit long-term bond prices and take small consideration to short-term prices.

4. A optimization model is used to obtain a new set of parameters \( b \) and then go back to step 2. The sequence of steps is repeated until the objective function \( G \) is minimized. In this thesis the mathematical program Matlab and its optimization method has been used to obtain the parameters.

### 5.3 Optimization of parameters

**Correlation**

\( \rho \) is the correlation between the interest rate and the stock price. It is difficult to decide this parameter. When pricing convertible bonds one is interested to know what the correlation will be in the future. It is possible to calculate the correlation on historical data but it is not sure that this is an acceptable approximation for the correlation in the future.

The correlation to be used when pricing convertible bonds is the correlation between the stock of the company and the interest rate of bonds issued by the same
company.
A reason implying that $\rho$ should be negative is that when the company is performing well the stock price will increase and the risk of the bonds will fall which causes the company interest rate to decrease. The opposite situation also implies a negative $\rho$ when the company is performing bad the stock price will fall and the increasing risk in the company will make the yield of company bonds to rise.
There are also mechanisms in the market causing positive correlation. If a big investor wants to change his investments from risky assets to assets with lower risk he may then sell stocks in company X and instead buy bonds issued by company X. An investor who wants to increase his risk and expected return may sell his possession of company X bonds and instead buy stocks in the same company. Both of these actions have are supposed to have an impact on the stock price and the interest rate and causing a positive correlation.

To estimate the correlation for a specific company interest rate and its stock it is preferable to have historical data for the short rate of the company. Normally the short rate for a company can not be found in the market. To get some insight of the correlation it is possible to look at the historical correlation between the stock and the shortest company interest rate that is available in the market.

For both Ericsson and Telia a graph of the historical correlation over the last 100 days have been constructed. The estimated correlation is between the stock price and the yield of the bond with the shortest time to maturity.

![Figure 5.1. Correlation between stock price and yield.](image)

In the Figure 5.1 it can be seen that the correlation is changing fast and therefore it is obvious that historical correlation is not a good estimate for the future correlation.
Since it is difficult to say anything of what correlation to use and historically the
correlation has sometimes been positive and sometimes negative the stock and inter-

est rate can be supposed to be uncorrelated i.e. \( \rho = 0 \).

**Government parameters**

To decide the mean reverting term \( a \) and the volatilities \( \sigma_{HW} \) in the Hull-White model and \( \sigma_{HL} \) in the Ho-Lee model one has to observe the market pricing of bond options. The market pricing in this thesis consists of prices on bond options for three different SGB:s (Swedish Government Bond), SGB1037, SGB1042 and SGB 1045. There are three different times to maturity 4, 8 and 12 months. For each underlying bond and maturity time there are three options with different strikes, one at the money, one in the money and one out of the money. The total number of SGB options is 27.

Optimizing SGB options with prices from the market gave values for \( a, \sigma_{HW} \) and \( \sigma_{HL} \) as in Table 5.1.

<table>
<thead>
<tr>
<th>Model</th>
<th>Company</th>
<th>( a_{\text{model}} )</th>
<th>( \sigma_{\text{model}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hull-White</td>
<td>SGB</td>
<td>0.1007</td>
<td>0.0134</td>
</tr>
<tr>
<td>Ho-Lee</td>
<td>SGB</td>
<td>0</td>
<td>0.0098</td>
</tr>
</tbody>
</table>

**Table 5.1.** Short rate parameters for government bonds

**Company parameters**

When pricing convertible bonds there are no CB:s on the Swedish market yet that uses SGB:s as an underlying bond, instead there is always an underlying company bond. A problem is that there are no bond options for this type of bonds in Sweden, not even an OTC market (Over The Counter). It is extremely difficult to find \( a \) without the use of bond options so a reasonable way to choose \( a \) is take to value from SGB option optimization.

A problem when trying to decide the Hull-White and the Ho-Lee parameters is that since there are no bond options for company bonds there is no way to derive the implicit interest rate volatility. In this thesis yield volatilities for the companies has been computed as historical yield volatilities.

Hull [3] shows that a connection between yield volatility and the zero coupon bond price volatility is

\[
\sigma_p = D y_0 \sigma_y. \tag{5.2}
\]

\( \sigma_p \) is the price volatility of a bond, \( D \) the duration of a bond, \( y_0 \) the initial forward yield and \( \sigma_y \) the yield volatility.

The price volatility for a ZCB with time to maturity \( T \) in the Hull-White and the Ho-Lee models are defined as

\[
\sigma_{p_{HW}} = \frac{\sigma_{HW}}{a} \left( 1 - e^{-aT} \right) \quad \sigma_{p_{HL}} = \sigma_{HL} T. \tag{5.3}
\]
Combining this equation with equation 5.2 and solving \( \sigma \) for both models gives

\[
Hull - White \quad \sigma_{HW} = \frac{Dy_0 \sigma_y a}{1 - e^{-aT}}
\]

\[
Ho - Lee \quad \sigma_{HL} = \frac{Dy_0 \sigma_y T}{T}
\]

Right side in the equation above are all known variables so solving that for the individual companies gave the result stated in Table 5.2.

<table>
<thead>
<tr>
<th>Model</th>
<th>Company</th>
<th>( a_{company} )</th>
<th>( \sigma_{company} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hull-White</td>
<td>Ericsson</td>
<td>0.101</td>
<td>0.0133</td>
</tr>
<tr>
<td>Ho-Lee</td>
<td>Ericsson</td>
<td>0</td>
<td>0.0126</td>
</tr>
<tr>
<td>Hull-White</td>
<td>Telia</td>
<td>0.101</td>
<td>0.00656</td>
</tr>
<tr>
<td>Ho-Lee</td>
<td>Telia</td>
<td>0</td>
<td>0.00639</td>
</tr>
</tbody>
</table>

Table 5.2. Short rate parameters for company bonds

The \( \sigma \) parameter for Telia under Hull-White and Ho-Lee is even lower than the governments \( \sigma \), this is due to the low yield volatility that Telia exhibits.

### 5.4 Parameter sensitivity

A study on the price of a convertible bond w.r.t. changes in different parameters reveals that different parameters have different influence of the price of the convertible bond. With the use of Hull-White short rate model as in equation 2.13 a fictive contract has been constructed with interesting convertible bond properties e.g. long time to maturity, reasonable length of American period and the parameters in the stochastic short rate model coming from section 5.3. By altering one parameter and letting the others be fixed one can observe the impact that the parameter have on the convertible bond price. The slope on the altering parameter graph is the Greek, see section 2.5, for that parameter.

The fictitious contract has a length of 5 years and parameters are given in Table 5.3. This collection of input are most realistic, a normal convertible bond has a

<table>
<thead>
<tr>
<th>stock price</th>
<th>conv. rate</th>
<th>short rate</th>
<th>( \sigma_{Stock} )</th>
<th>( \rho )</th>
<th>( a_{HW} )</th>
<th>( \sigma_{HW} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>45</td>
<td>4.25%</td>
<td>0.534</td>
<td>0</td>
<td>0.101</td>
<td>0.0133</td>
</tr>
</tbody>
</table>

Table 5.3. Contract parameters

maturity length of about 5 years with coupons every year and a normal length of
the American period is 2 months. The stock price volatility $\sigma_{\text{Stock}}$, $a_{\text{HW}}$ and $\sigma_{\text{HW}}$ for this fictive contract are all taken from the market and can be seen as real values. The stock price at time $t = 0$ where set to 45 and the interest rate to 0.0425.

**Parameters**

When altering stock price one can observe that the payoff from the convertible bond has a typical “hook look” just like ordinary options do. If the maturity had been shorter the “hook look” would be more distinguishable. When altering rate

![Graph showing price sensitivity](image)

**Figure 5.2.** Upper graph shows sensitivity in price when altering stock. Lower graph shows price sensitivity when altering short rate

the price exhibits negative correlation w.r.t. increasing rate. Although rates in the area of $r = [0.07, 0.10]$ the effect on the price of the convertible bond is not that drastic.

Having the right stock volatility for convertible bonds with long time to maturity is of great importance. In a sense it can be seen as a drawback to assume constant
volatility during the long time period but stochastic volatility is not the topic in this thesis.

The parameter $a_{HW}$ coming from bond option optimization can be seen as a rather unreliable estimate, looking at Figure 5.3 reveals the sensitivity in the parameter. The price on the convertible bond is not very sensitive for the parameter $a$ coming from the Hull-White model. By altering $a$ from 0 to 0.1 the price has just declined $\approx 0.12$ SEK which is about 0.2% of the price.

Altering $\sigma_{HW}$ has a large effect on the price of the convertible bond. Making a parallel shift of the forward rate curve up or down 1% results in a price change to $\pm 0.20$ SEK. Note that the short rate for today is not changed.

Figure 5.3. Upper graph shows sensitivity in price when altering stock volatility. Lower graph shows price sensitivity when altering $a_{HW}$. 
Figure 5.4. Upper graph shows sensitivity in price when altering rate volatility. Lower graph shows price sensitivity when making parallel shift in the forward rate curve.

It is common to assume that stock and short rate is uncorrelated e.g. $\rho = 0$. By not assuming that it is possible to observe if that assumption changes the price in a significant way. If the correlation is total, $\rho = -1$ or $\rho = 1$ there is a difference in price but observing $\rho$ in a more realistic interval $-0.3 \leq \rho \leq 0.3$ the difference in convertible bond price between correlation or not is $\approx 0.6\%$. 
Figure 5.5. Sensitivity in price when altering correlation factor.
Chapter 6

Results

At an initial stage of this work it was intended to price existing convertible bonds on the market. To price an existing contract one need to have bond data from that company. It has been difficult to get that kind of data. For two companies the data received were adequate i.e. more then two bonds. Unfortunately the convertible bonds for those companies where rather uninteresting, one being deep in the money with its price $\approx nS$ and the other far out of the money with price being just the bond. All data, estimates and results that has been used was generated at 2002-03-19.

6.1 Contracts

For this thesis two different convertible bonds have been constructed with different contract terms. A comparison between the Hull-White, Ho-Lee and the deterministic short rate model price has been done. The contract conditions are shown in Table 6.1. There are various other variables needed in each contract and they are stated in Table 6.2.

<table>
<thead>
<tr>
<th>Contract</th>
<th>Ericsson</th>
<th>Telia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maturity</td>
<td>070501</td>
<td>040501</td>
</tr>
<tr>
<td>Last day of conv.</td>
<td>070401</td>
<td>040401</td>
</tr>
<tr>
<td>Early conv.</td>
<td>070201</td>
<td>prior div.</td>
</tr>
<tr>
<td>Coupon</td>
<td>5%</td>
<td>4%</td>
</tr>
<tr>
<td>Face value</td>
<td>50</td>
<td>35</td>
</tr>
<tr>
<td>Conv. rate</td>
<td>50</td>
<td>35</td>
</tr>
</tbody>
</table>

Table 6.1. Contract conditions

Computing the convertible bond price will render a surface of prices for different stock prices and interest rates. The interesting area to observe is around the con-
version rate, observing at high stock prices the price is just \( nS \) and for low stock prices just the bond price.

### 6.2 Result contract Telia

This contract has a short time to maturity so there were reasons to believe that the choice of model should be not be so relevant. Figure 6.1 shows the differences between the Hull-White and the Ho-Lee model using the Telia contract from Table 6.1. According to the figure the differences in price are small when the short rate is close to the current short rate of 4.03%.

Pricing differences are bigger when the short rate is far from the current short rate. This is because the mean reversion term \( a \) in the Hull-White model will have an effect on the future interest rate development. There may be a small difference in

<table>
<thead>
<tr>
<th>Table 6.2. Parameters in the models</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contract</td>
</tr>
<tr>
<td>( \sigma_{Hull-White} )</td>
</tr>
<tr>
<td>( \sigma_{Hull-White} )</td>
</tr>
<tr>
<td>( \sigma_{Ho-Lee} )</td>
</tr>
<tr>
<td>( \sigma_{Stock} )</td>
</tr>
<tr>
<td>Dividend</td>
</tr>
<tr>
<td>( \rho )</td>
</tr>
<tr>
<td>( r_0 )</td>
</tr>
</tbody>
</table>

![Figure 6.1. Difference surface btw Hull-White and Ho-Lee. The price using Hull-White is 43.88 at S=35 and r=4%](image)
the pricing of the coupons too. When the stock price is higher the price difference goes to zero. That is because for high stock prices the convertible bond price tends to $\approx nS$.

Figure 6.2 shows the differences at a more narrow interval in short rate i.e. around $r_0$ and stock price. It is quite obvious that the pricing differences are small for changes in the stock price and bigger for changes in the interest rate. This is because the different models will handle interest rate changes differently.
Taking difference between the Hull-White and deterministic Ho-Lee the graphs will have the same appearance and shape however the sizes of the price differences are slightly different. It can be seen in appendix A. The difference using Ho-Lee and deterministic Ho-Lee is in this case relatively small. None of the models have a mean reversion term. That has the effect that the expected interest rate development will follow a curve parallel to the expected forward rate. Since the pricing difference is small the pricing differences will be small, that can be seen in the Figure 6.3.

In appendix B several of the Greeks are shown as surfaces for the Hull-White model.

6.3 Result contract Ericsson

This contract has a long time to maturity so there was reason to believe that the choice of model should be relevant. Figure 6.4 shows the differences in the Hull-White and Ho-Lee model using the Ericsson contract from Table 6.1. For this contract too the pricing differences are close to zero when the interest rate is close to the current rate. However the price differences are of a bigger magnitude than for the Telia contract. Figure 6.5 shows the differences at a more narrow interval in short rate i.e. around $r_0$ and stock price. It can be seen that the price difference is of magnitude $\pm 0.10 \ SEK$ for changes in the short rate which is about 0.3% of the price.

The difference between Hull-White and Ho-Lee deterministic can be seen in appendix A, the shape of the difference graph is basically the same as for the Hull-White and Ho-Lee comparison. The difference between Ho-Lee and Ho-Lee deterministic is still small although somewhat larger than in the Telia case due to longer

Figure 6.4. Difference surface btw Hull-White and Ho-Lee. The price using Hull-White is 66.89 at $S=42$ and $r=4.25\%$
Figure 6.5. Difference btw Hull-White and Ho-Lee around current short rate and stock price. The price using Hull-White is 66.89 at \( S=42 \) and \( r=4.25\% \)

Figure 6.6. Difference btw Ho-Lee and Ho-Lee deterministic. The price using Ho-Lee 66.90 at \( S=42 \) and \( r=4.25\% \)

maturity time.

In appendix C several of the Greeks are shown as surfaces for the Hull-White model.
6.4 Conclusion and summary

A comparison between Ho-Lee and the deterministic model shows that the pricing differences are very small. There is a more significant pricing difference between Hull-White and the other models. The conclusion is that introducing a stochastic term to the short rate will only have a small effect on convertible bond price. When also including mean reversion the pricing differences are more significant. The results show that a longer time to maturity will make the choice of short rate model more important.

Even though using different short models has a small impact on the convertible bond pricing the choice of model can be of greater importance when hedging convertible bonds. This is because hedging has to take into account what effect interest rate changes will have on the price and the short rate models all handle interest rate changes differently.

For risky convertible bonds the use of a good default risk model might be more important than the choice of short rate model. An interesting extension to this thesis could be to introduce a good default risk model for pricing both risky and not so risky convertible bonds.

Since the time to maturity for convertible bonds often are long the use of constant stock price volatility is not realistic. To achieve a better model for convertible bond pricing the use of stochastic stock price volatility could be useful.
Appendix A

Hull-White and Ho-Lee deterministic

Figure A.1. Difference surface between Hull-White and deterministic Ho-Lee for the Telia contract.
Figure A.2. Difference surface between Hull–White and deterministic Ho–Lee for the Ericsson contract.
Appendix B

Telia greek surface

Figure B.1. $\frac{\partial V}{\partial \rho}$ for Telia contract.
Figure B.2. $\frac{dV}{dr}$ for Telia contract.

Figure B.3. Delta for Telia contract.
Figure B.4. Gamma for Telia contract.

Figure B.5. Theta for Telia contract.
Figure B.6. Vega for Telia contract.
Appendix C

Ericsson greek surface

Figure C.1. $\frac{\partial V}{\partial \rho}$ for Ericsson contract.
Figure C.2. $\frac{dV}{dr}$ for Ericsson contract.

Figure C.3. Delta for Ericsson contract.
**Figure C.4.** Gamma for Ericsson contract.

**Figure C.5.** Theta for Ericsson contract.
Figure C.6. Vega for Ericsson contract.
Bibliography


