Proof of the minimax theorem

**Lemma:** The theorem of separating hyperplanes

- Let $M$ be a convex, closed set in $\mathbb{R}^n$
- Let $\bar{x}$ be a point where $\bar{x} \notin M$

Then there exists a vector $\bar{c}$ and a number $k$ so that $\bar{c} \cdot \bar{x} = k$ and $\bar{c} \cdot \bar{y} > k$ for all $\bar{y} \in M$.

Given an $m \times n$ game matrix $A$:

$$A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & \ddots & \cdots & \vdots \\
  \vdots & \ddots & \ddots & \vdots \\
  a_{m1} & \cdots & a_{mn} & \ddots & \vdots \\
  \vdots & \ddots & \cdots & \cdots & \ddots \\
  \end{pmatrix}$$

Where:

- Player I’s objective: find $\bar{x}$ so that all components of $\bar{x}A$ are $\geq V_{\min}$.
- Player II’s objective: find $\bar{y}$ so that all components of $A\bar{y}^\top$ are $\leq V_{\max}$.

The minimax theorem states that $V_{\min} = V_{\max}$. To prove its correctness, we will first show that $V_{\min} < 0 < V_{\max}$ is impossible. Let:

$$A_0 = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\
  a_{21} & \ddots & \cdots & 0 & 1 & \ddots & \cdots & 0 \\
  \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  a_{m1} & \cdots & a_{nn} & \ddots & \cdots & \ddots & \cdots & 1 \\
  \vdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \\
  \end{pmatrix}_{n+m}$$

Now, let $B$ be the convex hull of the columns in $A_0$. We have two cases:

1. $\bar{0} \in B$.
2. $\bar{0} \notin B$. 

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1) \( \bar{0} \in B \quad (\bar{0} \in \mathbb{R}^m) \)

Then there are \( z_1, z_2, \ldots, z_n, z_{n+1}, \ldots, z_{n+m} \) so that \( 0 \leq z_i \leq 1, \sum z_i = 1 \) and \( \sum_{j=1}^n a_{ij} z_j + \sum_{k=1}^n \delta_{ik} z_{n+k} = 0 \), which gives us \( \sum_{j=1}^n a_{ij} z_j \leq 0 \) for all \( i \).

It is impossible for \( z_j = 0 \) to be true for all \( j \leq n \), and thus \( \sum_{j=1}^n z_j > 0 \).

Now, let \( y_j = \frac{z_j}{\sum z_j}, \quad 1 \leq j \leq n \). Then \( \sum_{j=1}^n a_{ij} y_j \leq 0 \) and \( y_j \) gives a mixed strategy. We can conclude that \( V_{\text{max}} \leq 0 \).

2) \( \bar{0} \notin B \)

Then there are \( c_1, c_2, \ldots, c_m, k \) so that \( \bar{c} \cdot \bar{0} = k \) (that is, \( k = 0 \)) and \( \bar{c} \cdot \{\text{column in } A_0\} > 0 \).

This gives us \( c_i > 0 \), for all \( i \), and \( \sum_{i=j}^m c_i a_{ij} > 0 \) for \( 1 \leq j \leq n \). Now, \( x_i = \frac{c_i}{\sum c_i} \Rightarrow \sum_{i=1}^m a_{ij} x_i > 0 \). Which illustrates that \( V_{\text{min}} > 0 \).

We have shown that \( V_{\text{min}} < 0 < V_{\text{max}} \) is impossible. Now we can use this result to show that \( V_{\text{min}} < t < V_{\text{max}} \) is impossible for all \( t \).

Given a game matrix \( A \), create \( A' \), equal to \( A \) with \( t \) subtracted from every component. If \( M(\bar{x}, \bar{y}) = \bar{x} A \bar{y}^T \) and \( M'(\bar{x}, \bar{y}) = \bar{x} A' \bar{y}^T \) holds, then \( M'(\bar{x}, \bar{y}) = M(\bar{x}, \bar{y}) - t \) and \( V_{\text{min}} < t < V_{\text{max}} \Rightarrow V_{\text{min}} < 0 < V_{\text{max}} \), which is impossible.

**Structure theorem**

Let \( \bar{x}^*, \bar{y}^* \) be optimal strategies. Strategies where \( x_i^* > 0 \) and \( y_j^* > 0 \) are called **active**. Let \( k_i = \sum_{j} a_{ij} y_j^*, l_j = \sum_{i} a_{ij} x_i^* \).

Now:
- if \( i \) is active, \( k_i = v \).
- if \( j \) is active, \( l_j = v \).

...where \( v \) is the payoff of the game.

**Proof**

We have \( M(\bar{x}^*, \bar{y}^*) = v, v = v \sum_i x_i^* \) and \( v = \sum_i k_i x_i^* \). This implies that \( \sum_i (v - k_i) x_i^* = 0 \).
Since $v - k_i \geq 0$, we get that $v - k_i = 0$ for all $x_i^* > 0$.

**Special case**

We have a special case if $A$ is an $n \times n$ matrix and all strategies are active. Assume $A$ is invertible, and let:

$$\bar{J} = (1, 1, \ldots, 1), \quad \bar{J}^T = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Player II is looking for a mixed strategy $\bar{y}$ so that:

$$A\bar{y}^T = v\bar{J}^T \Rightarrow \bar{y}^T = vA^{-1}\bar{J}^T$$

We must have $\bar{J} \cdot \bar{y}^T = 1$ which gives the sum of the matrix components:

$$vJA^{-1}\bar{J}^T = 1 \Rightarrow v = \frac{1}{JA^{-1}\bar{J}}$$

The resulting mixed strategies are:

$$\bar{y}^T = \frac{A^{-1}\bar{J}^T}{JA^{-1}\bar{J}}, \quad \bar{x} = \frac{\bar{J}A^{-1}}{JA^{-1}\bar{J}}$$

**If $A$ is not invertible**

Assume that $A$ is not necessarily invertible. We define $A^*$ by the relation $AA^* = |A| \cdot I$. (if $|A| \neq 0$ use $A^{-1} = \frac{A^*}{|A|}$).

Assume that $\bar{J}A^*\bar{J}^T \neq 0$. We can then show that $v = \frac{|A|}{\bar{J}A^*\bar{J}^T}$ and that:

$$\bar{x} = \frac{\bar{J}A^*}{JA^*\bar{J}^T}, \quad \bar{y}^T = \frac{A^*\bar{J}^T}{JA^*\bar{J}^T}$$

**The other way around**

Assume that we have an $n \times n$ matrix $A$, and assume that $\bar{J}A^*\bar{J}^T \neq 0$ and that $\frac{\bar{J}A^*\bar{J}^T}{JA^*\bar{J}^T}$, $\frac{\bar{J}A^*}{JA^*\bar{J}^T}$ are vectors with all components $> 0$.

Then there exist optimal mixed strategies:

$$\bar{x} = \frac{\bar{J}A^*}{JA^*\bar{J}^T}, \quad \bar{y}^T = \frac{A^*\bar{J}^T}{JA^*\bar{J}^T}$$

and the game’s payoff is:

$$v = \frac{|A|}{\bar{J}A^*\bar{J}^T}$$
Example

We have a game matrix:

\[ A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha > 0, \beta > 0 \]

This kind of game is usually called “attacking a hidden object”, where \( \alpha \) is the probability that the object is destroyed by a successful attack. Where should the object be hidden?

\[ A^* = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \]

\[ \bar{J}A^*J^T = \alpha + \beta, \quad |A| = \alpha\beta \]

And the game’s payoff is the harmonic mean of \( \alpha \) and \( \beta \):

\[ v = \frac{\alpha\beta}{\alpha + \beta} \]

\[ \bar{x} = \left( \frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \right), \quad \bar{y}^T = \left( \frac{\beta}{\alpha + \beta}, \frac{\alpha}{\alpha + \beta} \right) \]

Extensive form

In the extensive form, a game is modelled as a tree, with states as nodes and the possible moves as edges. The leaves of the tree represent the possible payoffs. See the example in figure ??.

Figur 1: Example of a game in the extensive form
Elements of chance do not break the extensive model; chance can be included as a “third player”, see figure ??.

If the game lacks elements of chance, and the payoff is either 1 (win) or 0 (loss), we have a purely combinatorial game.

The extensive form is not useful for modeling all kinds of games, however. Examples of games that can’t be modelled completely using the extensive form:

- games where the players pick their moves simultaneously,
- games where the same position can occur an infinite number of times.

**Information sets**

With the addition of information sets, we can use the extensive form to model games where the players do not always have complete information.

Let’s say we have a game where the two players I and II pick a number 0 or 1 independently of each other. If they pick the same number, I wins, if they pick different numbers, II wins.

Due to the tree structure used in the extensive form, one of the players has to pick first, giving the other player the freedom of picking just the right move to win. But with the introduction of information sets, we can specify that a number of game states are to be viewed as one and the same state from the active player’s point of view. See the example in figure ?? where the two states in the cloud are the same as far as player II knows.

**Complete game model**

A game is modelled as a finite tree $\Gamma$, where:

- every node is labeled I, II, S, or is a leaf.
- no child has the same label as it’s parent.
Figur 3: Example of an information set

- every leaf is labeled with a number (the payoff for player I), or, for non-zero-sum games, a vector of the payoffs for all players.
- the I-nodes are partitioned into information sets.
- there is no downward path from an information set into itself.
- all nodes in the same information set have the same degree $k$. Every node has it’s edges labeled with $k$ symbols $s_1, s_2, \ldots, s_k$.
- the same applies to the II-nodes.